The Sixth Overseas Chinese Physics Association

ACCELERATOR SCHOOL

Beijing, China, July 29 - August 7, 2010

Lattice Design

C.C. Kuo
郭錦城

July 29 ~ August 7, 2010
Beijing

Some of the materials, figures, etc., are from references:

1. S.Y. Lee, Accelerator Physics, World Scientific, 2nd edition, 2004.
2. Helmut Wiedemann, Particle Accelerator Physics, Springer, 3rd edition, 2007.
3. Klause Wille, The Physics of Particle Accelerators, Oxford University Press, 2000
4. A. W. Chao and Maury Tigner, Handbook of Accelerator Physics and Engineering, World Scientific, 3rd edition, 1999.
5. And other resources, public web pages, etc.

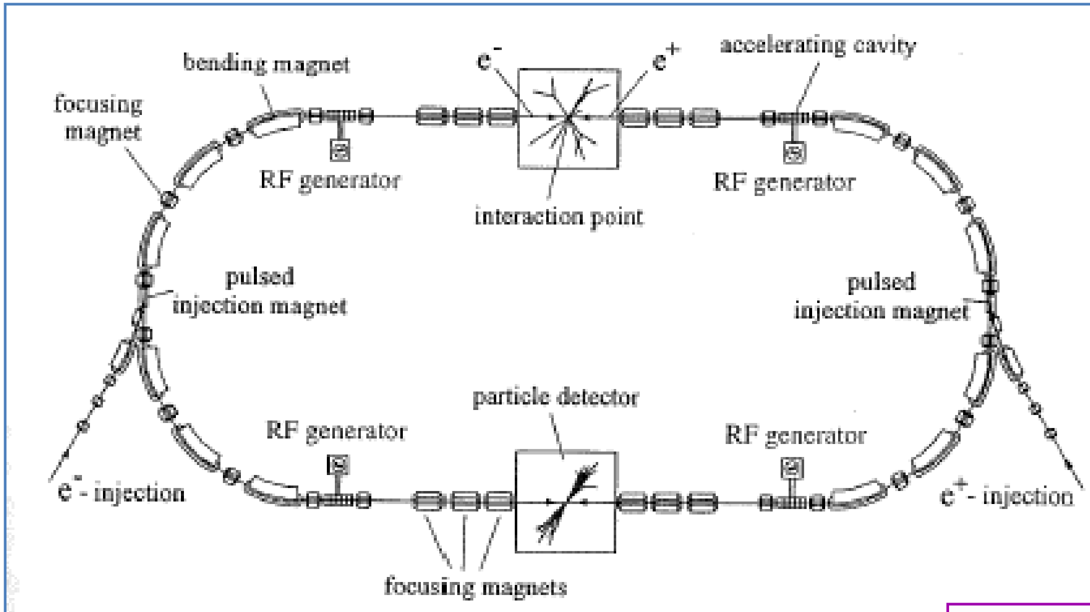
These lecture notes are only for this School.

outline

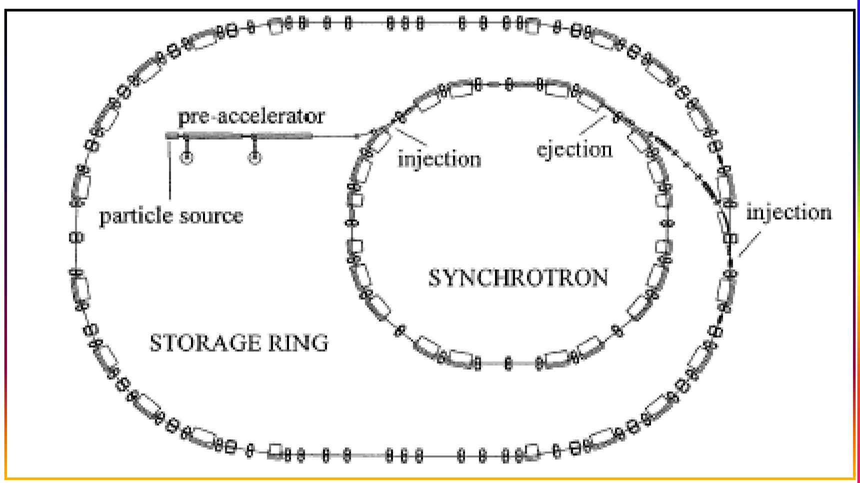
- Introduction
- Linear beam dynamics
- Linear lattice
- Nonlinear beam dynamics
- Examples
- Imperfections, corrections and lifetime

Accelerator Lattice Design

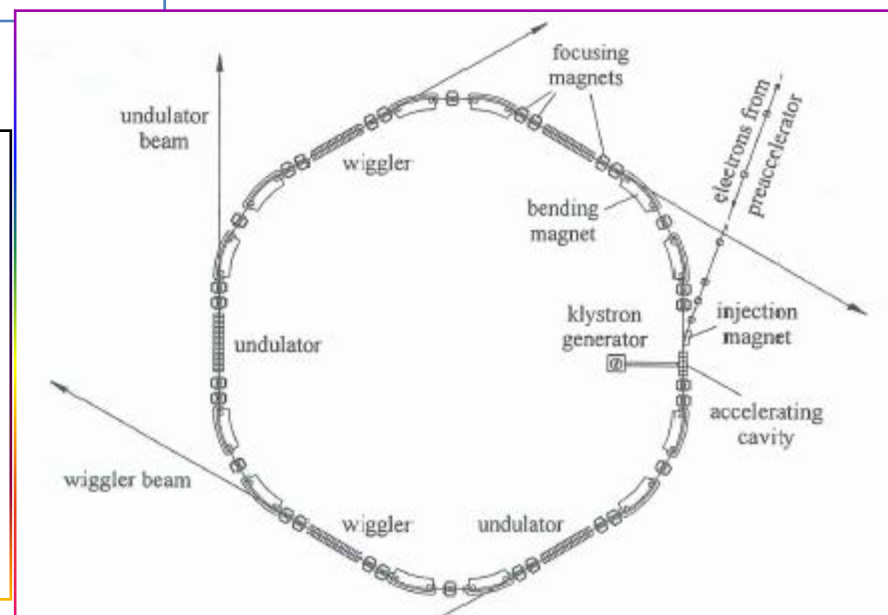
- The arrangement of the magnets along beam path for guiding or focusing charged particles is called the magnet lattice or lattice. The arrangement can be irregular array or repetitive regular array of magnets.
- The repetitive regular array is called periodic lattice.
- In a circular accelerator, the periodic lattice can be symmetric. Usually, the lattice is constructed from cells and then superperiods. A number of superperiods then complete a ring.
- The goal of lattice design is to obtain simple, reliable, flexible, and high performance accelerators that meet users' request.
- For modern circular accelerators, lattice design is of crucial importance in the design of the accelerators.



collider

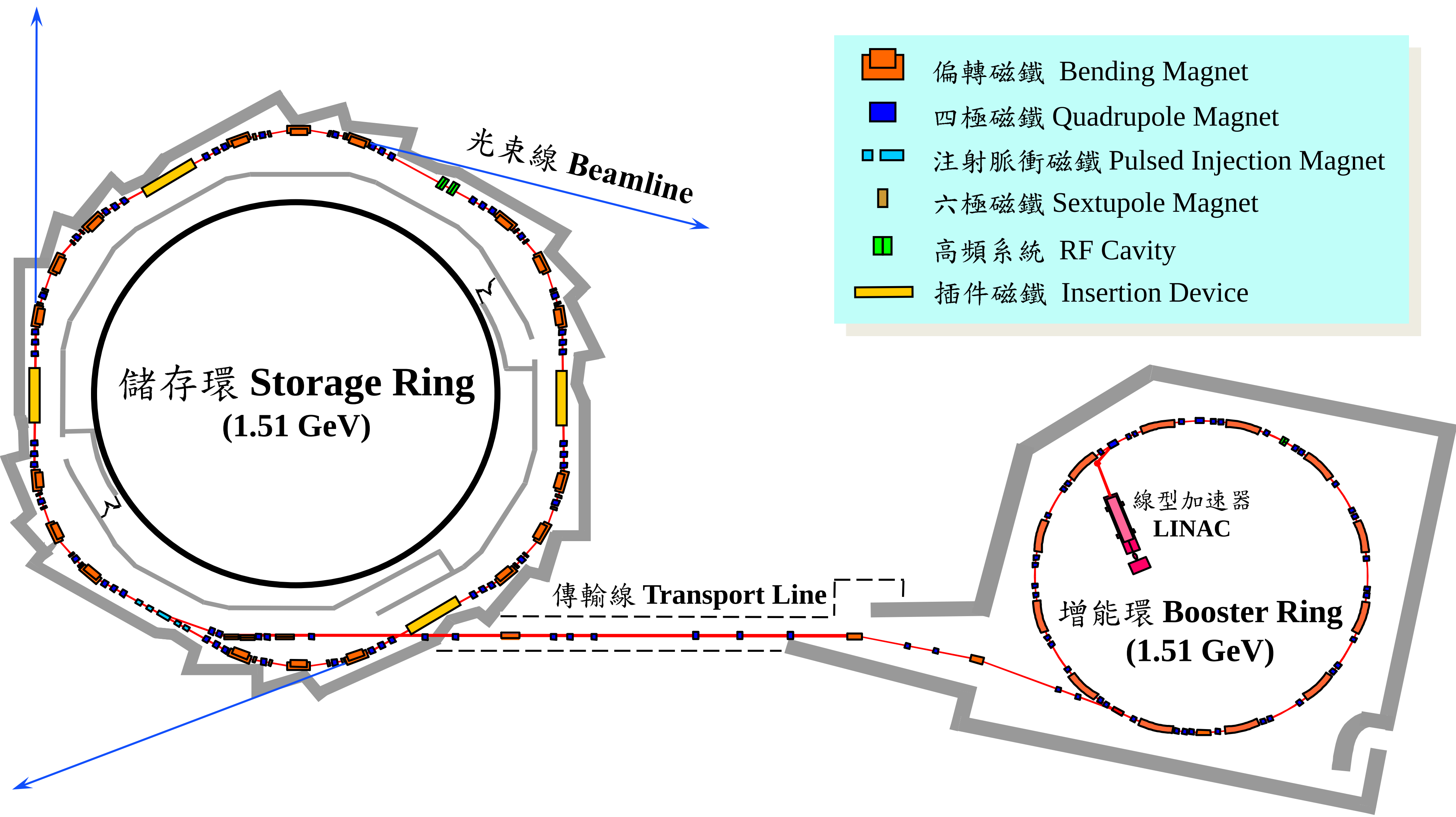


Accelerator chain



Synchrotron light source

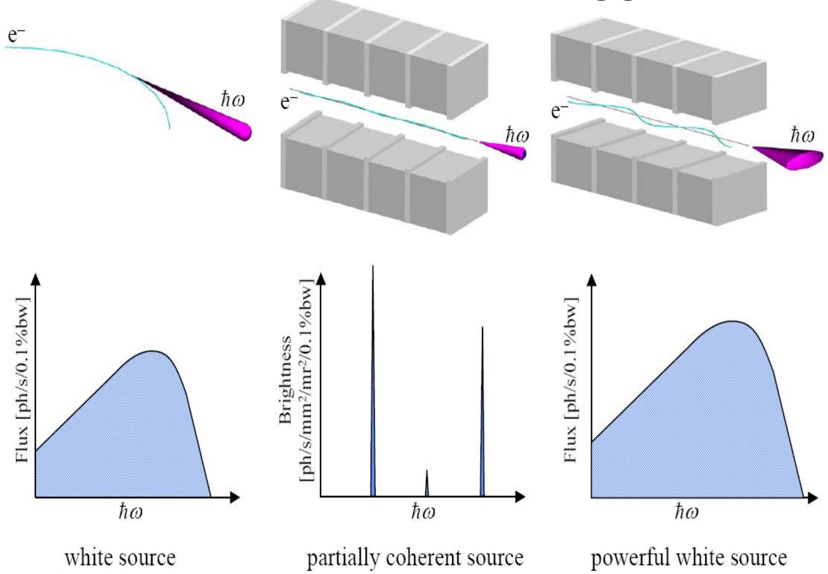
NSRRC TLS

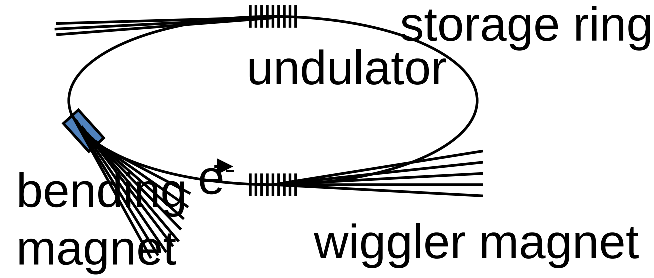


偏轉磁鐵 Bending Magnet
四極磁鐵 Quadrupole Magnet
注射脈衝磁鐵 Pulsed Injection Magnet
六極磁鐵 Sextupole Magnet
高頻系統 RF Cavity
插件磁鐵 Insertion Device
光束線 Beamline
儲存環 Storage Ring
(1.51 GeV)
傳輸線 Transport Line
線型加速器
LINAC
增能環 Booster Ring
(1.51 GeV)

Synchrotron Light Source

Bend Undulator Wiggler





Critical energy : $\varepsilon_c[\mathrm{KeV}] = 0.665 B[\mathrm{T}] E^2[\mathrm{GeV}]$

Critical wavelength : $\lambda_c(\mathring{A}) = 5.59 \dfrac{\rho[m]}{E^3[GeV]}$

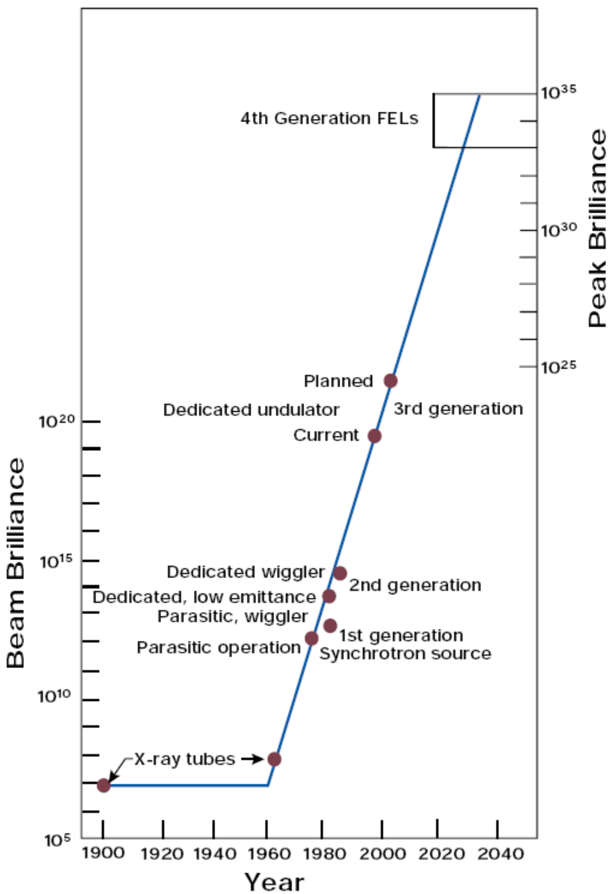
Photon flux : $\dfrac{dN}{d\theta}\Big|_{\lambda=\lambda_c} \left[\dfrac{\text{photons}}{\text{amp sec mrad } \theta \text{ } 0.1\%\text{BW}}\right] = 1.6\times 10^{13}\,\mathrm{E[GeV]}$

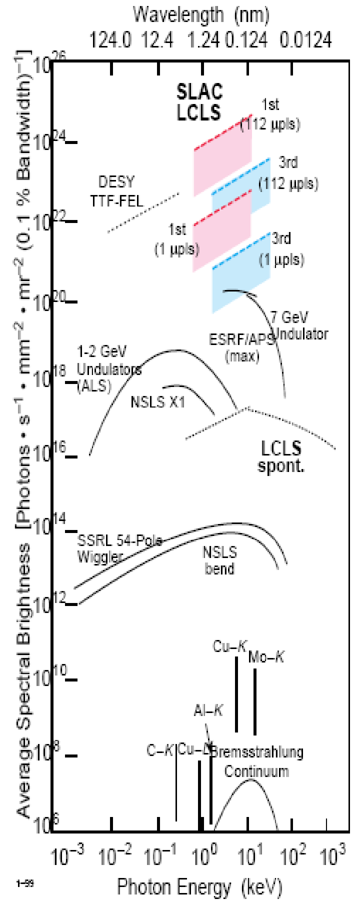
Radiation per turn : $\mathrm{U_0[KeV]} = \dfrac{88.5\mathrm{E}^4[\mathrm{GeV}]}{\rho[\mathrm{m}]}, \quad E[\mathrm{GeV}] = 0.3B[\mathrm{T}]\rho[\mathrm{m}]$

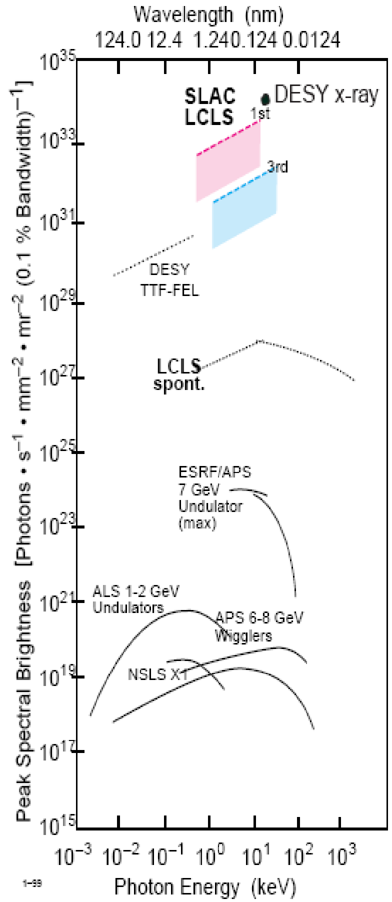
Beam Brilliance of Light Source

$$B = \frac{dN/dt}{4\pi\sigma_x\sigma_{x'}\sigma_y\sigma_{y'}\frac{\Delta\omega}{\omega}} \propto \frac{I}{\kappa\varepsilon_x^2}$$

[photon/sec/mm²/mrad²/0.1%bandwidth]







Luminosity in Collider

$$L = \frac{N_+ N_-}{4\pi\sigma_x\sigma_z} f_{collision} = \frac{N_+ N_-}{4\pi\sigma_x\sigma_z} f_0 n_b$$

Due to Coulomb force of opposite beam,

beam-beam tune-shift due to quadrupole-like force :

$$\xi_{z\pm} = \frac{N_\mp r_0 \beta_z^*}{2\pi\gamma\sigma_z(\sigma_x+\sigma_z)}, \xi_{x\pm} = \frac{N_\mp r_0 \beta_x^*}{2\pi\gamma\sigma_x(\sigma_x+\sigma_z)}$$

β_x^*, β_z^* are betatron function at interactio n point (IP)

for $\sigma_x >> \sigma_z$, $\xi_z \approx 0.05 \sim 0.1$

$N_+ = N_-$

$$L \approx \frac{\pi\gamma^2}{r_0^2}\frac{\sigma_x\sigma_z}{\beta_x^{*2}}\xi_z^2 f_{collision}$$

Optimization of luminosity, emittance can not be too small. The design of e^+e^- colliders differ substantially from synchrotron light sources,

Lattice Design Procedure

1) For a modern accelerator, lattice design work usually takes some years to finalize the design parameters. It is an iterative process, involving users, funding, accelerator physics, accelerator subsystems, civil engineering, etc.
2) It starts from major parameters such as energy, size, etc.
3) Then linear lattice is constructed based on the building blocks. Linear lattice should fulfill accelerator physics criteria and provide global quantities such as circumference, emittance, betatron tunes, magnet strengths, and some other machine parameters.

Lattice Design Procedure

4) Design codes such as MAD are used for the lattice functions matching and parameters calculations.
5) Usually, a design with periodic cells is needed in a circular machine. The cell can be FODO, Double Bend Achromat (DBA), Triple Bend Achromat (TBA), Quadruple Bend Achromat (QBA), or Multi-Bend Achromat (MBA or nBA) types.

① Combined-function or separated-function magnets are selected.

② Maximum magnetic field strengths are constrained. (room- temperature or superconducting magnets, bore radius or chamber profile, etc.)

③ Matching or insertion sections are matched to get desired machine functions.

Lattice Design Procedure

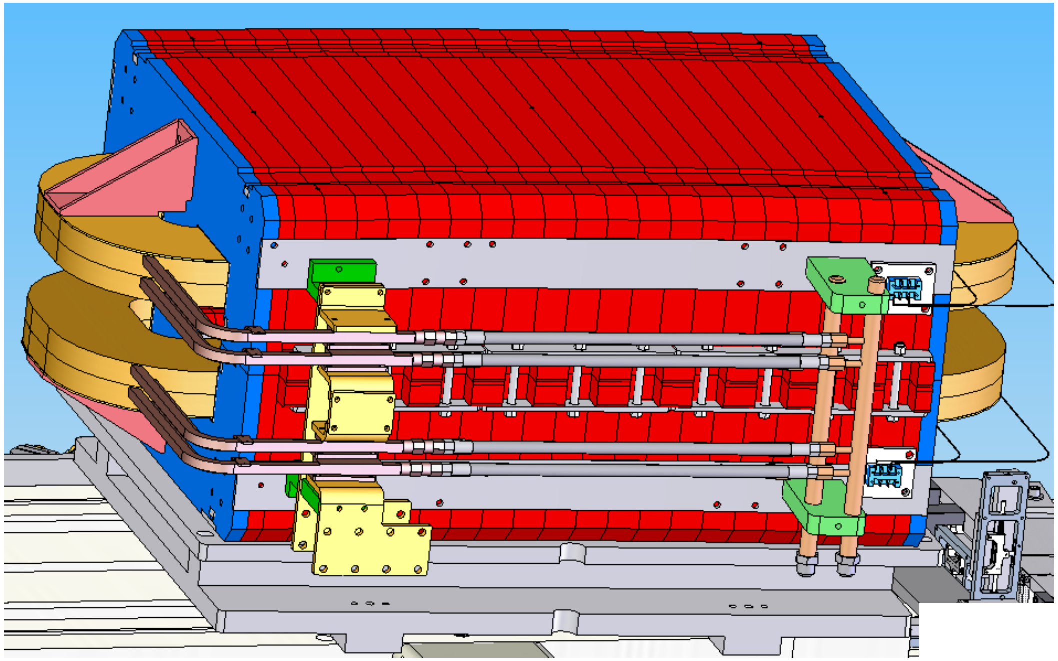
6) To get stable solution of the off-momentum particle, we need to put sextupole magnets and RF cavities in the lattice beam line. Such nonlinear elements induce nonlinear beam dynamics and the dynamic acceptances in the transverse and longitudinal planes need to be carefully studied in order to get sufficient acceptances. (for long beam current lifetime and high injection efficiency)
7) For the modern high performance machines, strong sextupole fields to correct high chromaticity will have large impact on the nonlinear beam dynamics and it is the most challenging and laborious work at this stage.

Lattice Design Procedure

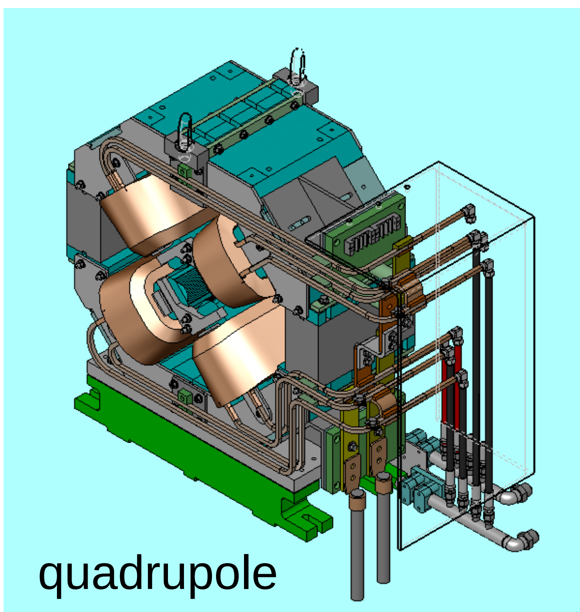
8) In the real machine, there are always imperfections in the accelerator elements. So, one need to consider engineering/alignment limitation or errors , vibrations, etc. Correction schemes such as orbit correction, coupling correction, etc., need to be developed. (dipole correctors, skew quadrupoles, beam position monitors, etc)
9) Make sure long enough beam current lifetime, e.g., Touschek lifeime, in the real machine including insertion devices, etc.
10) To achieve a successful accelerator, we need to consider not only lattice design but many issues, which might be covered in this school.

Lattice Design Process

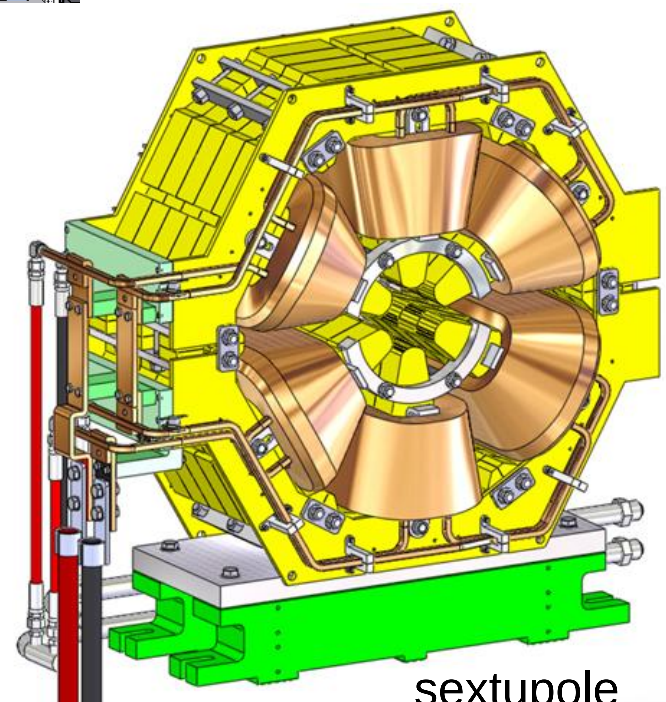
1) User requirements
2) Lattice type (DBA, TBA, QBA, nBA… FODO)
3) Linear lattice
4) Nonlinear dynamic aperture tracking
5) Longitudinal dynamics
6) Error tolerance analysis
7) Satisfying user requirements (if NO, go back to step 2)
8) ID effects, lifetime, COD, coupling, orbit stability, instabilities, injection, etc.



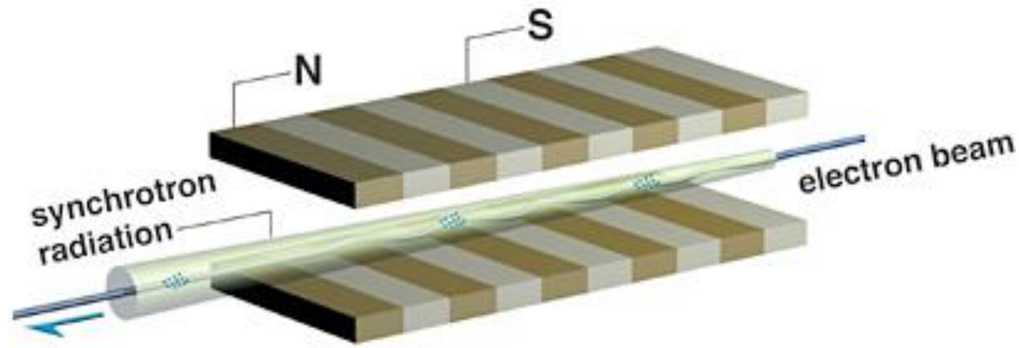
dipole

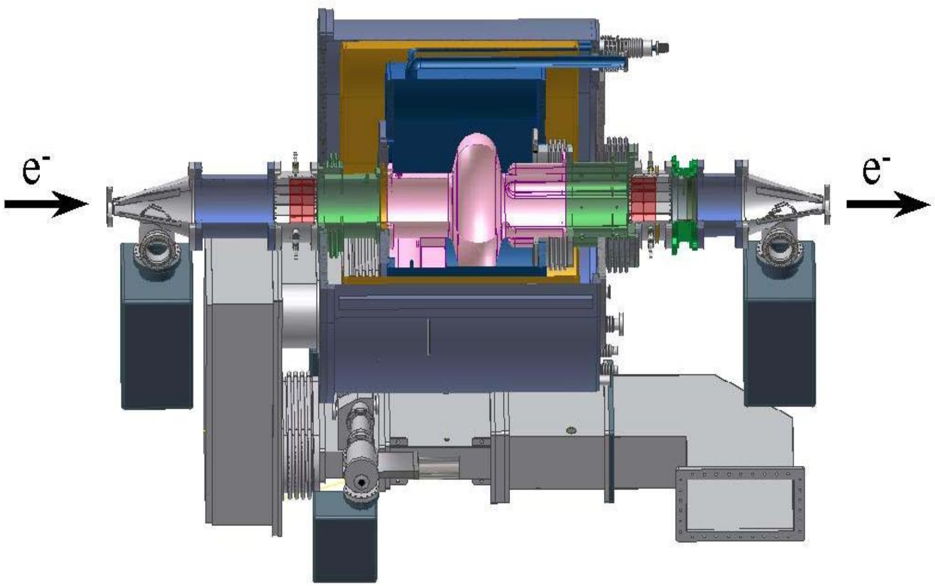


quadrupole



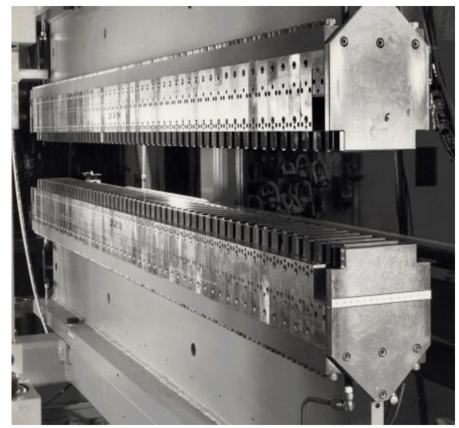
sextupole



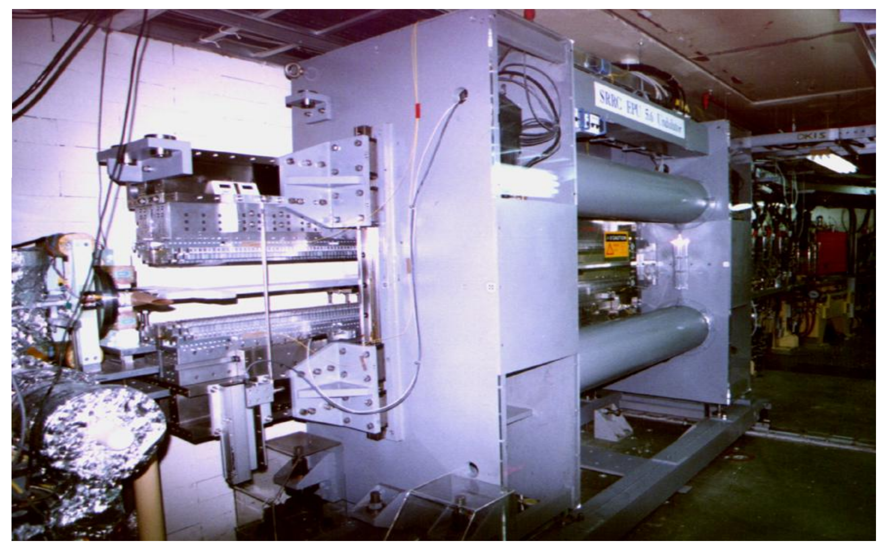


SC RF

(Beam Acceleration)



Insertion devices



Equation of motion

$$\frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v} \times \vec{B})$$

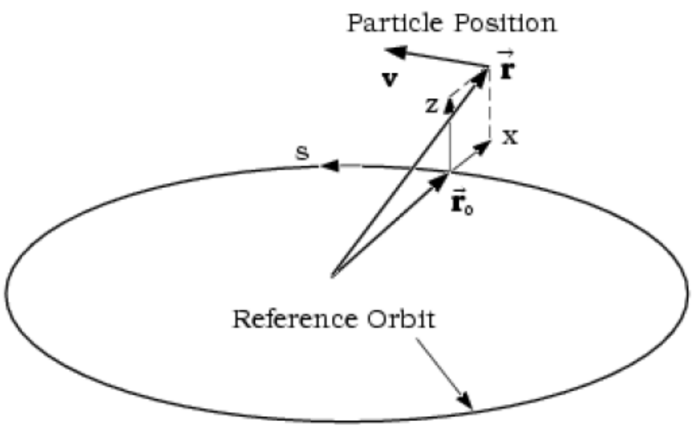
$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$H = e\Phi + c\sqrt{m^2c^2 + (\vec{p} - e\vec{A})^2}$$

Frenet - Serret Coordinate system :

$$H = -p(1 + \frac{x}{\rho}) + \frac{1 + x/\rho}{2p}[(p_x - eA_x)^2 + (p_z - eA_z)^2] - eA_s$$



$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho}$$

Hill's equation

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s)$$

$$\phi_{n+1} = \phi_n + \frac{2\pi\eta}{\beta^2 E}\Delta E_{n+1}$$

Synchrotron motion

Equation of motion

Betatron equation of motion :

$$\begin{cases} x'' - \dfrac{\rho + x}{\rho^2} = \pm \dfrac{B_z}{B\rho} \dfrac{p_0}{p} (1 + \dfrac{x}{\rho})^2 \\ z'' = \mp \dfrac{B_x}{B\rho} (1 + \dfrac{x}{\rho})^2 \end{cases}$$

$$B_z + jB_x = B_0 \sum_{n=0}^{\infty} (b_n + ja_n)(x + jz)^n,$$

$$b_n = \frac{1}{B_0 n!} \frac{\partial^n B_z}{\partial x^n} |_{x=z=0}, \quad a_n = \frac{1}{B_0 n!} \frac{\partial^n B_x}{\partial x^n} |_{x=z=0}$$

b_0 : normal dipole $\quad B_z = B_0 b_0$

a_0 : skew dipole $\quad B_x = B_0 a_0$

b_1 : normal quad $\quad B_z = B_0 b_1 x, \ B_x = B_0 b_1 z$

b_1 : skew quad $\quad B_z = -B_0 a_1 z, \ B_x = B_0 a_1 x$

b_2 : normal sext $\quad B_z = B_0 b_2 (x^2 - z^2), \ B_x = 2B_0 b_2 xz$

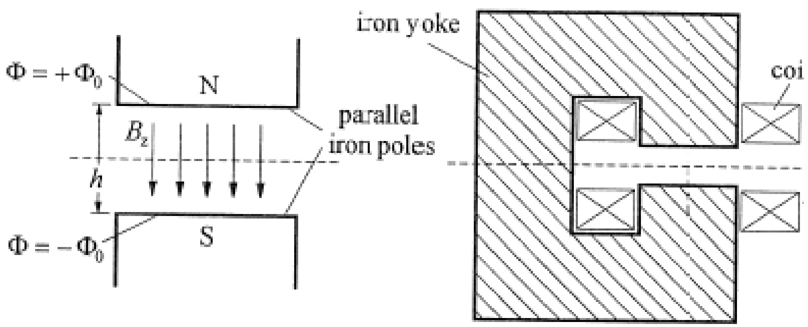
b_2 : skew sext $\quad B_z = -2B_0 a_1 xz, \ B_x = B_0 a_2 (x^2 - z^2)$

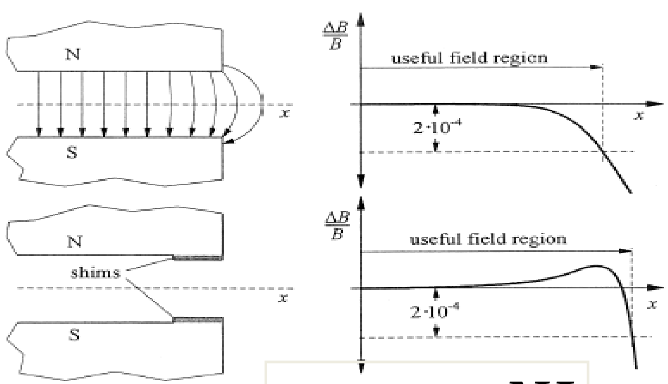
Hill's equation:

$$\begin{cases} x'' + K_x(s) x = \dfrac{\Delta B_z}{B\rho} \\ z'' + K_z(s) z = -\dfrac{\Delta B_x}{B\rho} \end{cases}$$

$$K_x(s) = \frac{1}{\rho^2} - \frac{1}{B\rho} \frac{\partial B_z}{\partial x}, \quad K_z(s) = \frac{1}{B\rho} \frac{\partial B_x}{\partial z},$$

Main magnets

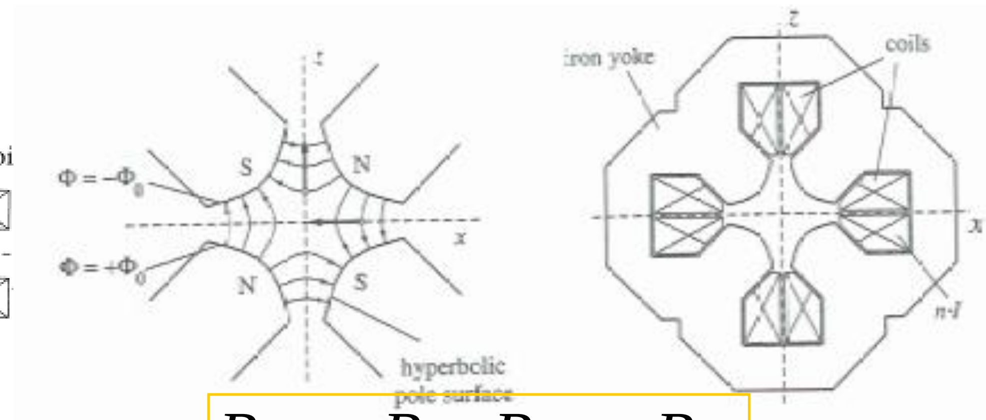




dipole

$$B = -\frac{\mu_0 NI}{h}$$

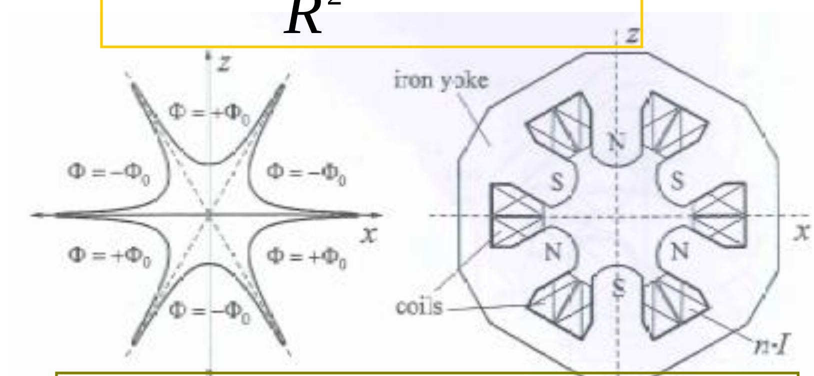
B field limits (example):
Dipole < 1.5T
Quad poletip < 0.7T
Sext poletip < 0.4T



$$B_x = -B_1 z,\ B_z = -B_1 x$$

$$B_1 = \frac{2\mu_0 NI}{R^2}$$

quadrupole

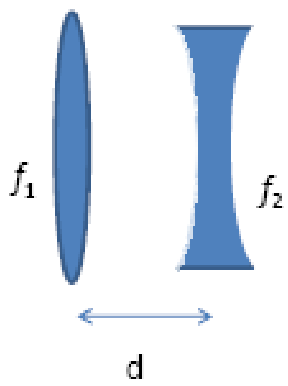


$$B_x = B_2 xz,\ B_z = \frac{1}{2} B_2 (x^2 - z^2)$$

$$B_2 = 6\mu_0 NI / R^3$$

sextupole

Focusing



$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} > 0$$

By properly arrange focusing and defocusing quadrupole magnets, the overall effects are focusing in both radial (horizontal) and vertical planes.

Combined function lattice: quadrupole field and/or sextupole fields in bending magnet.
Separated function lattice: no quadrupole or sextupole fields in bending magnet.

Ideal accelerator (linear motion):

$$x'' + K_x(s)x = 0, \quad z'' + K_z(s)z = 0, \quad K_x(s) = \frac{1}{\rho^2} - \frac{1}{B\rho}\frac{\partial B_z}{\partial x}, \quad K_z(s) = +\frac{1}{B\rho}\frac{\partial B_x}{\partial z},$$

Matrix formalism in linear beam dynamics:

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = M(s,s_0)\begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

(1) Focusing quadrupole

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix}, K > 0, \ \ell = s - s_0$$

$$= \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \text{for thin-lens approximation, } \ell \to 0, \ f\,(\text{focal length}) = \lim_{\ell \to 0} \frac{1}{|K|\ell}$$

(2) Defocusing quadrupole

$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix}, \ K < 0, \ell = s - s_0$$

$$= \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}, \text{for thin-lens approximation}$$

(3) Drift space K=0

$$M(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}, \ell = s - s_0$$

(4) pure sector dipole: $K_x = \frac{1}{\rho^2}$, $M_x(s,s_0) = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}, \theta = \ell/\rho$

$M_x(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$ for small $\theta = \ell/\rho$

non-deflecting plane: $M_z(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$

(5) pure rectangular dipole due to wedge in both ends:

In deflecting plane: $M_x(s,s_0) = \begin{pmatrix} 1 & 0 \\ 1/f_x & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f_x & 1 \end{pmatrix}$

$$M_x(s,s_0) = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \text{ where } \frac{1}{f_x} = \frac{1}{\rho}\tan(\frac{\theta}{2})$$

In non-deflecting plane: $M_z(s,s_0) = \begin{pmatrix} 1 & 0 \\ -1/f_z & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_z & 1 \end{pmatrix}$,

$$\text{where } \frac{1}{f_z} = \frac{1}{\rho}(1 - \frac{2G}{3L})\tan(\frac{\theta}{2})$$

G = half pole gap, L = straight magnet length

Beam dynamics in transport line

- In open transport lines the phase space can be transferred using transfer matrix piecewise. We need initial condition.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{sn} = M(s_n, s_{n-1})\ldots M(s_3, s_2) M(s_2, s_1) \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

- In terms of Courant-Snyder parameters, there are relations between initial and final points along the beam path.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) \begin{pmatrix} x \\ x' \end{pmatrix}_{s1} \qquad M(s_2, s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi + \alpha_1 \sin\psi) & \sqrt{\beta_1\beta_2}\sin\psi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\psi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi - \alpha_2 \sin\psi) \end{pmatrix}$$

Beam Dynamics in Periodic Lattice

Hill's equation:

$$y'' + K(s)y = 0, \quad K(s+L) = K(s)$$

Floquet's theorm for the solution of Hill's equation:

(1) Two independent solutions

$$y_1(s) = w(s)e^{i\mu(s/L)}, \; y_2(s) = w^*(s)e^{-i\mu(s/L)},$$

$w^*(s)$: complex conjugate to $w(s)$

and $w(s+L) = w(s)$

(2) $\cos\mu = \frac{1}{2}\mathrm{Tr}(M)$

(3) $\det M = 1$

(4) $-2 \le \mathrm{Tr}(M) \le 2$

Courant –Snyder Parameters for periodic lattice

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = I\cos\Phi + J\sin\Phi$$

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \text{Trace}(J) = 0, J^2 = -I \Rightarrow \beta\gamma = 1 + \alpha^2$$

$$\text{eigenvalues } \lambda = e^{\pm i\Phi} = \cos\Phi \pm i\sin\Phi$$

$$\Phi = \cos^{-1}[(M_{11} + M_{22})/2]$$

$$\beta = M_{12}/\sin\Phi$$

$$\alpha = (M_{11} - M_{22})/2\sin\Phi$$

where tranfer matrix is for one period or one turn.

Using similarity transformation $M(s_2) = M(s_2, s_1)M(s_1)M(s_2, s_1)^{-1}$

for any two points in the beam line

We have

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{22} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

Using Froquet transformation

We get $y(s) = a\sqrt{\beta}e^{\pm i\psi}, \psi(s-s_0) = \int_{s_0}^{s}\frac{d\varsigma}{\beta(\varsigma)}, \Phi = \int_0^L \frac{ds}{\beta(s)}$

And

$$M(s_2,s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi + \alpha_1\sin\psi) & \sqrt{\beta_1\beta_2}\sin\psi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\psi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi - \alpha_2\sin\psi) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix}$$

Stability of FODO Cell and Optimum Phase Advance Per Cell

FODO cell : $\frac{1}{2}QF-L-QD-L-\frac{1}{2}QF,\ f=1/k\ell, \ell=$ half quad length

$$M_{FODO}=\begin{pmatrix}1-2\frac{L^2}{f^2} & 2L(1+\frac{L}{f}) \\ -1/f^* & 1-2\frac{L^2}{f^2}\end{pmatrix} \text{ with } f_f=f_d=f,\ 1/f^*=2(1-L/f)(L/f^2)$$

We can get :

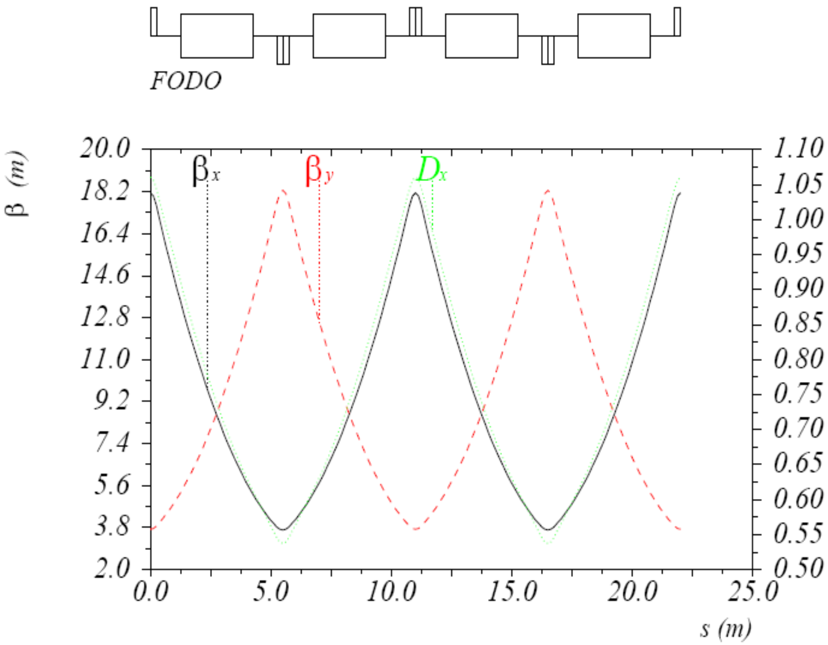
$$\beta_x(\text{middle of QF})=\beta_x^{+}=L\frac{\kappa(\kappa+1)}{\sqrt{\kappa^2-1}} \text{ with } \kappa=f/L \Rightarrow \kappa>1 \rightarrow f>L$$

$$\beta_x(\text{middle of QD})=\beta_x^{-}=L\frac{\kappa(\kappa-1)}{\sqrt{\kappa^2-1}}$$

Maximum betatron function can be minimized
with optimum phase advance per FODO cell :

$$\frac{d\beta_x^{\pm}}{d\kappa}=0 \Rightarrow \phi_0=76.345^{\circ}$$

Exercise 1



FODO cell : $\frac{1}{2}QF - L - QD - L - \frac{1}{2}QF$, $f = 1/k\ell, \ell =$ half quad length

and $\frac{1}{2}QD - L - QF - L - \frac{1}{2}QD$

$$M_{FODO} = \begin{pmatrix} 1 & 0 \\ \mp 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \pm 2/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mp 1/f & 1 \end{pmatrix}$$

$$M_{FODO} = \begin{pmatrix} 1-2\frac{L^2}{f^2} & 2L(1\pm\frac{L}{f}) \\ -1/f^* & 1-2\frac{L^2}{f^2} \end{pmatrix} \text{ with } f_f = f_d = f,\ 1/f^* = 2(1\mp L/f)(L/f^2)$$

$$= \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi + \alpha\sin\Phi \end{pmatrix}$$

$$\Rightarrow \cos\Phi = 1 - 2\frac{L^2}{f},\ \sin\frac{\Phi}{2} = \frac{L}{f}, \beta^{\pm} = 2L\frac{(1\pm\sin\frac{\Phi}{2})}{\sin\Phi}, \alpha = 0$$

```
L1:DRIFT,L=1
L2:DRIFT,L=1
QFH     :QUADRUPOLE,L=.5/2, K1=0.5
QDH     :QUADRUPOLE,L=.5/2, K1=-0.5
BD      :SBEND,L=3,ANGLE=TWOPI/96
HSUP    :LINE=(QFH,L1,BD,L2,QDH)
FSUP    :LINE=(HSUP,-HSUP)
FODO    :LINE=(2*FSUP)
RING    :LINE=(48*FSUP,)
```

Betatron Tune $\nu_{x,z} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,z}(s)}$

General solution:

$$y(s) = a\sqrt{\beta}\cos(\psi + \psi_0)$$

Normalized coordinate $w = \frac{y}{\sqrt{\beta}}, \phi = \frac{1}{\nu}\int^{s} \frac{d\varsigma}{\beta(\varsigma)} = \frac{\psi}{\nu}$

$$w(\phi) = a\cos\nu(\phi + \phi_0)$$

$$\frac{d^2 w}{d\phi^2} + \nu^2 w = 0$$

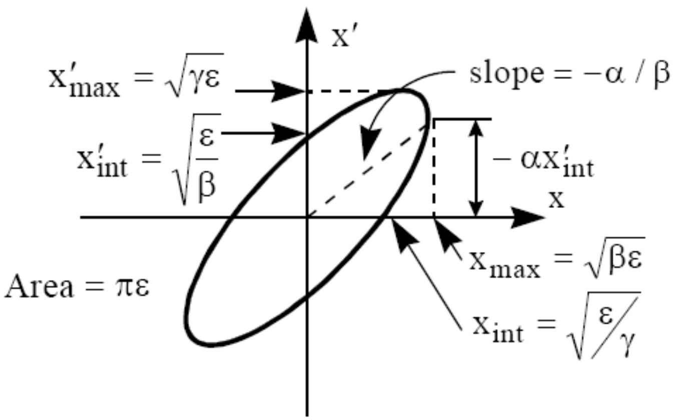
$\Rightarrow$ a simple harmonic oscillation in normalized coordinate.

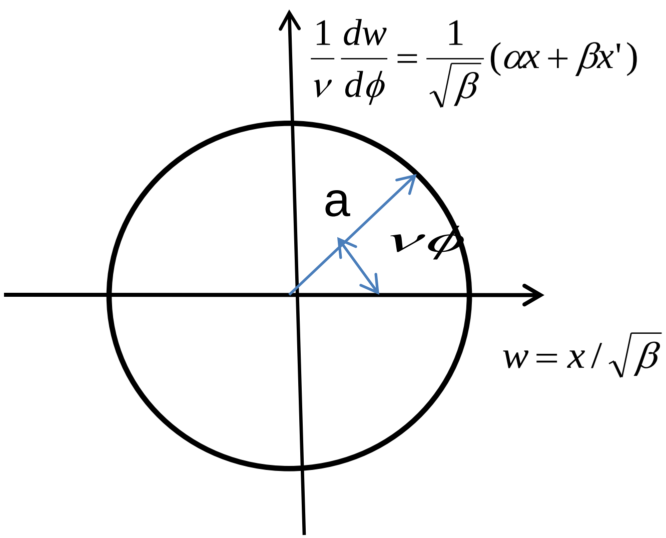
Courant – Snyder Invariant and Emittance

$$C(y, y') = \frac{1}{\beta}(y^2 + (\alpha y + \beta y')^2) = \gamma y^2 + 2\alpha y y' + \beta y'^2 = \varepsilon = \text{emittance} = 2J = a^2$$

Beam size : $\sigma(s) = \sqrt{\varepsilon\beta(s) + (D(s)\sigma_\delta)^2}$

Beam divergence : $\sigma'(s) = \sqrt{\varepsilon\gamma(s) + (D'(s)\sigma_\delta)^2}$





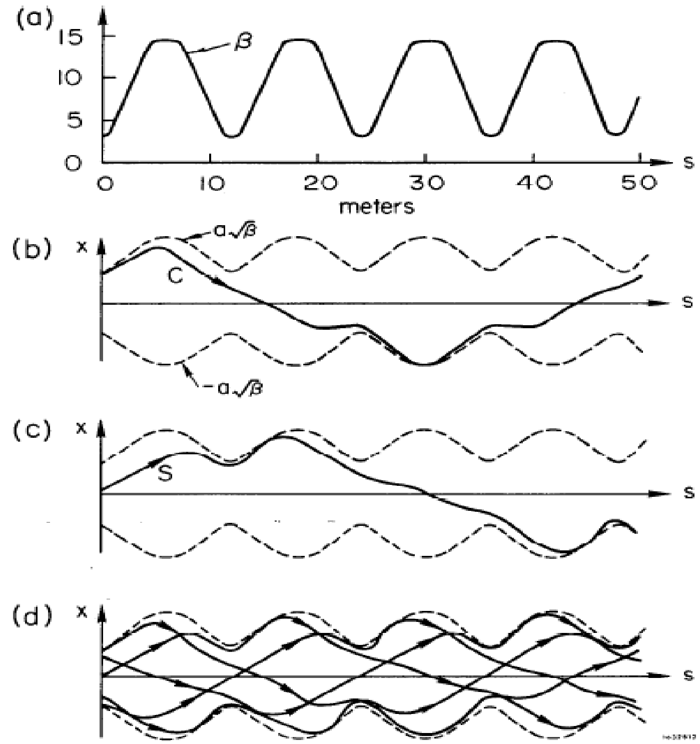


FIG. 12--(a) Betatron function. (b) Cosine-like trajectory for s=0. (c) Sine-like trajectory for s=0. (d) One trajectory on several successive revolutions.

M. Sands SLAC-121

Off-momentum orbit

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{k_x(s)}{1+\delta}\right)x = \frac{\delta}{\rho(1+\delta)}, k_x(s) = \frac{1}{B\rho}\frac{\partial B_z}{\partial x}$$

Let $x = x_\beta(s) + D(s)\delta$

then $x_\beta'' + (K_x(s) + \Delta K_x)x_\beta = 0$

$$D'' + (K_x(s) + \Delta K_x)D = \frac{1}{\rho} + o(\delta)$$

where $K_x(s) = \frac{1}{\rho^2} - k_x(s), \Delta K_x = [-\frac{2}{\rho^2} + k_x(s)]\delta + o(\delta^2)$

To the lowest order in $\delta \Rightarrow D'' + K_x(s)D = \frac{1}{\rho}$

$D(s+L) = D(s),\ D'(s+L) = D'(s)$

$$\begin{pmatrix} D(s) \\ D'(s) \end{pmatrix} = M(s,s_0)\begin{pmatrix} D(s_0) \\ D'(s_0) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix} \quad 2\times 2 \text{ matrix}$$

$$\begin{pmatrix} d \\ d' \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho K_x}(1-\cos\sqrt{K_x}s) \\ \frac{1}{\rho\sqrt{K_x}}\sin\sqrt{K_x}s \end{pmatrix} \quad \text{if } K_x > 0,$$

$$\begin{pmatrix} d \\ d' \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho|K_x|}(-1+\cosh\sqrt{|K_x|}s) \\ \frac{1}{\rho\sqrt{|K_x|}}\sinh\sqrt{|K_x|}s \end{pmatrix} \quad \text{if } K_x < 0$$

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = M\begin{pmatrix} D(s_0) \\ D'(s_0) \\ 1 \end{pmatrix} \quad 3\times 3 \text{ matrix}$$

For pure sector dipole:

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -(1/\rho)\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$M \overset{\theta\to 0}{=} \begin{pmatrix} 1 & \ell & \ell\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

For pure rectangular dipole:

$$M = \begin{pmatrix} 1 & \rho\sin\theta & \rho(1-\cos\theta) \\ 0 & 1 & 2\tan\frac{\theta}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

For pure quadrupole:

$$M = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell & 0 \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion for periodic lattice

- For periodic lattice, the closed-orbit condition:

$$\begin{pmatrix} \mathrm{D} \\ \mathrm{D'} \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{D} \\ \mathrm{D'} \\ 1 \end{pmatrix}$$

$$D = \frac{M_{13}(1 - M_{22}) + M_{12} M_{23}}{2 - M_{11} - M_{22}}$$

$$D' = \frac{M_{13} M_{21} + (1 - M_{11}) M_{23}}{2 - M_{11} - M_{22}}$$

Solving M_{13}, M_{23}

$$M = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi & (1 - \cos\Phi - \alpha\sin\Phi)D - \beta\sin\Phi\, D' \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi & \gamma\sin\Phi\, D + (1 - \cos\Phi + \alpha\sin\Phi)\, D' \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion in a FODO Cell

For one half FODO cell, 1/2 QF - B - 1/2 QD :

$$M_{1/2FODO} = \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L^2/(2\rho) \\ 0 & 1 & L/\rho \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-L/f & L & L^2/(2\rho) \\ -L/f^2 & 1+L/f & (L/\rho)(1+L/2f) \\ 0 & 0 & 1 \end{pmatrix}$$

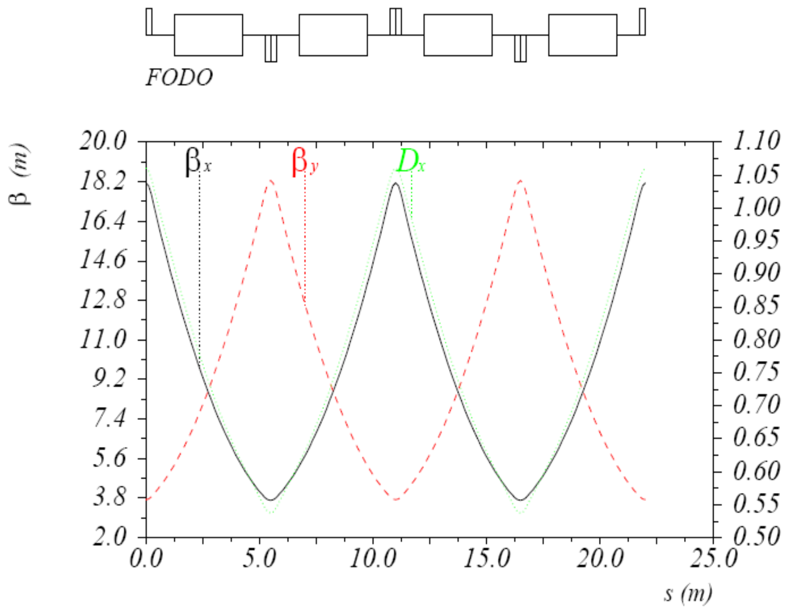
Transformation through one half FODO cell :

$$\begin{pmatrix} D^- \\ 0 \\ 1 \end{pmatrix} = M_{1/2FODO} \begin{pmatrix} D^+ \\ 0 \\ 1 \end{pmatrix}$$

with $D' = 0$ in the middle of quads, we get

$$D^+ = \frac{f^2}{\rho}\left(1 + \frac{L}{2f}\right)$$

$$D^- = \frac{f^2}{\rho}\left(1 - \frac{L}{2f}\right)$$



FODO Cell

For full FODO, f is the focal length of quadrupole, L is the half cell length, Φis the phase advance of the FODO cell.

$$\sin\frac{\Phi}{2}=\frac{L}{2f},\beta_{F,D}=\frac{2L(1\pm\sin(\Phi/2))}{\sin\Phi},\alpha_{F,D}=0$$

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}=M_{FODO}\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

$$D_F=\frac{L\theta(1+\frac{1}{2}\sin(\Phi/2))}{\sin^2(\Phi/2)},D_F'=0$$

$$D_d=\frac{L\theta(1-\frac{1}{2}\sin(\Phi/2))}{\sin^2(\Phi/2)},D_d'=0$$

Integral Representation of Dispersion Function

Same as dipole field error representation, we can substitute

$$\frac{\Delta B}{B_0\rho} \to \frac{\Delta p}{p_0}\frac{1}{\rho}, \quad x_{co} \to D(s)\frac{\Delta p}{p_0}$$

$$D(s) = \frac{\sqrt{\beta_x(s)}}{2\sin\pi\nu_x}\int_s^{s+C}\frac{\sqrt{\beta_x(t)}}{\rho}\cos(\pi\nu_x - |\psi_x(t)-\psi_x(s)|)dt$$

Normalized dispersion function :

$$X_d(s) = \frac{1}{2\sin\pi\nu_x}\int_s^{s+C}\frac{\sqrt{\beta_x(t)}}{\rho}\cos(\pi\nu_x - |\psi_x(t)-\psi_x(s)|)dt$$

$$X_d(s) = \sqrt{2J_d}\cos\Phi_d$$

$$P_d(s) = \frac{-1}{2\sin\pi\nu_x}\int_s^{s+C}\frac{\sqrt{\beta_x(t)}}{\rho}\sin(\pi\nu_x - |\psi_x(t)-\psi_x(s)|)dt$$

$$P_d(s) = -\sqrt{2J_d}\sin\Phi_d$$

$$J_d = \frac{1}{2}H(D,D') = \frac{1}{2}\frac{1}{\beta_x}[D^2 + (\beta_x D' + \alpha_x D)^2]$$

Achromatic Lattice

- In principle, dispersion can be suppressed by one focusing quadrupole and one bending magnet.
- With one focusing quad in between two dipoles, one can get achromat condition. In between two bends, we call it arc section. Outside arc section, we can match dispersion to zero. This is so called double bend achromat (DBA) structure.
- We need quads outside arc section to match the betatron functions, tunes, etc.
- Similarly, one can design triple bend achromat (TBA), quadruple bend achromat (QBA), and multi-bend achromat (MBA or nBA) structure.
- For FODO cells structure, dispersion suppression section at both ends of the standard cells.

DBA

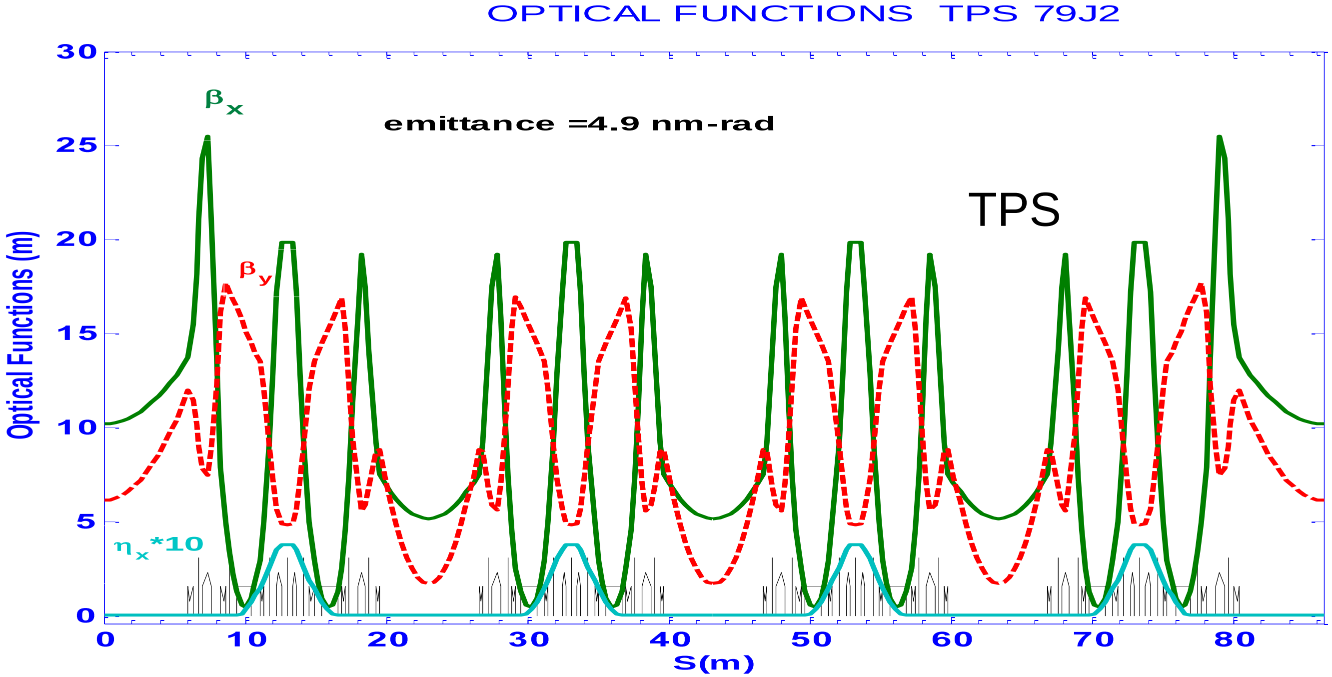
Consider a simple DBA cell with a single quadrupole in the middle. In thin-lens approximation, the dispersion matching condition :

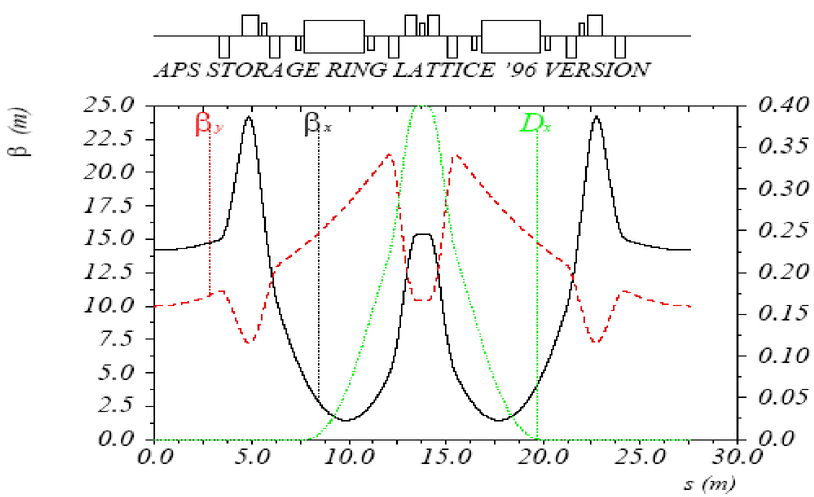
$$\begin{pmatrix} D_c \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/(2f) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

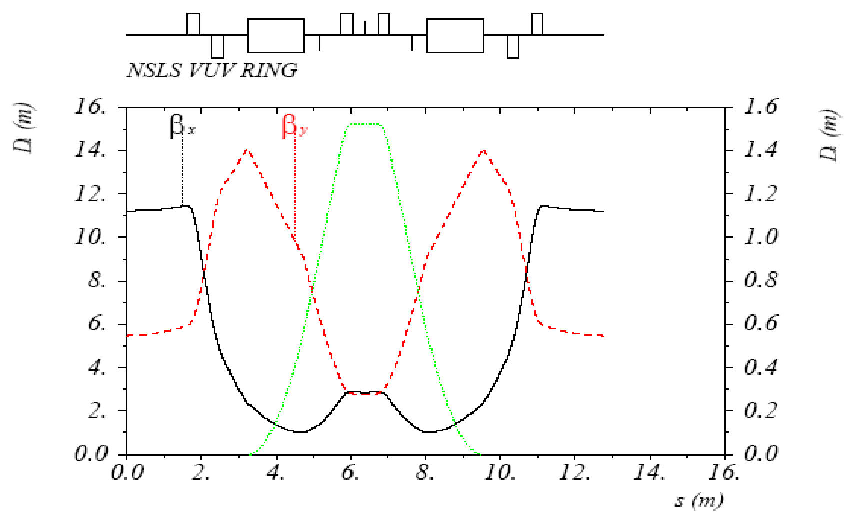
Where *f* is the focal length of quad, θ and L are the bending angle and dipole length, L_1 is the distance between end of dipole to center of quad.

$$f = \frac{1}{2}(L_1 + \frac{1}{2}L), \quad D_c = (L_1 + \frac{1}{2}L)\theta$$

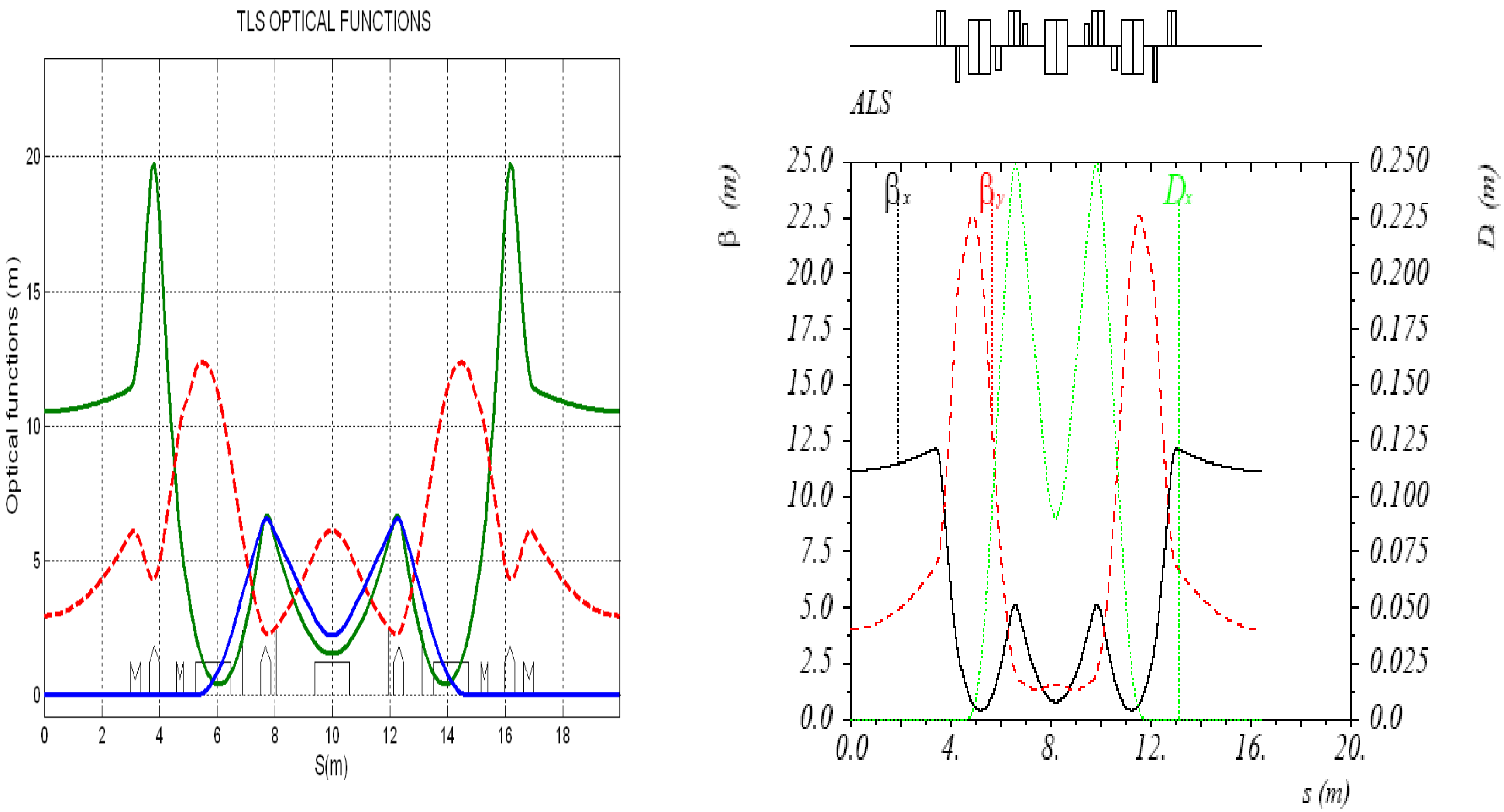
DBA


OPTICAL FUNCTIONS TPS 79J2
emittance =4.9 nm-rad
TPS
βx
βy
ηx*10
Optical Functions (m)
S(m)
0
5
10
15
20
25
30
0
10
20
30
40
50
60
70
80


APS STORAGE RING LATTICE '96 VERSION
β (m)
βy
βx
Dx
s (m)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
22.5
25.0
0.0
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.0
5.0
10.0
15.0
20.0
25.0
30.0


NSLS VUV RING
Ω (m)
D (m)
βx
βy
s (m)
0.0
2.
4.
6.
8.
10.
12.
14.
16.
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
0.0
2.
4.
6.
8.
10.
12.
14.
16.

TBA

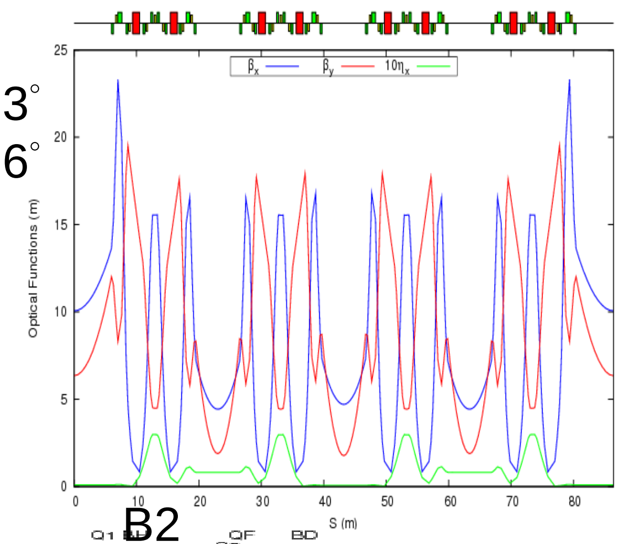


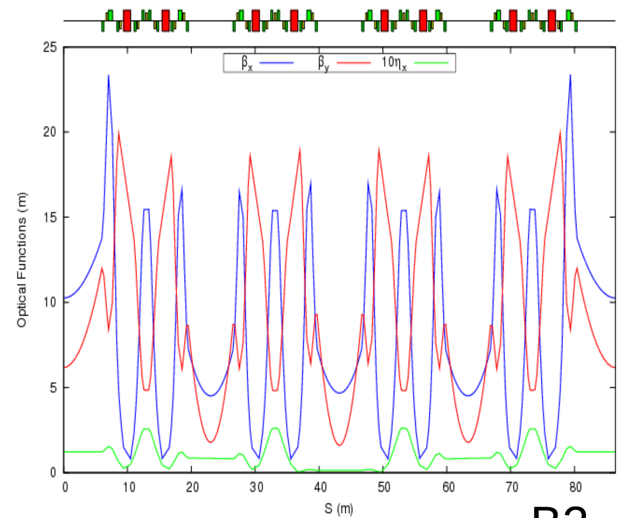
Combined function dipoles

QBA, OBA, and nBA

B2 B1 B1 B2

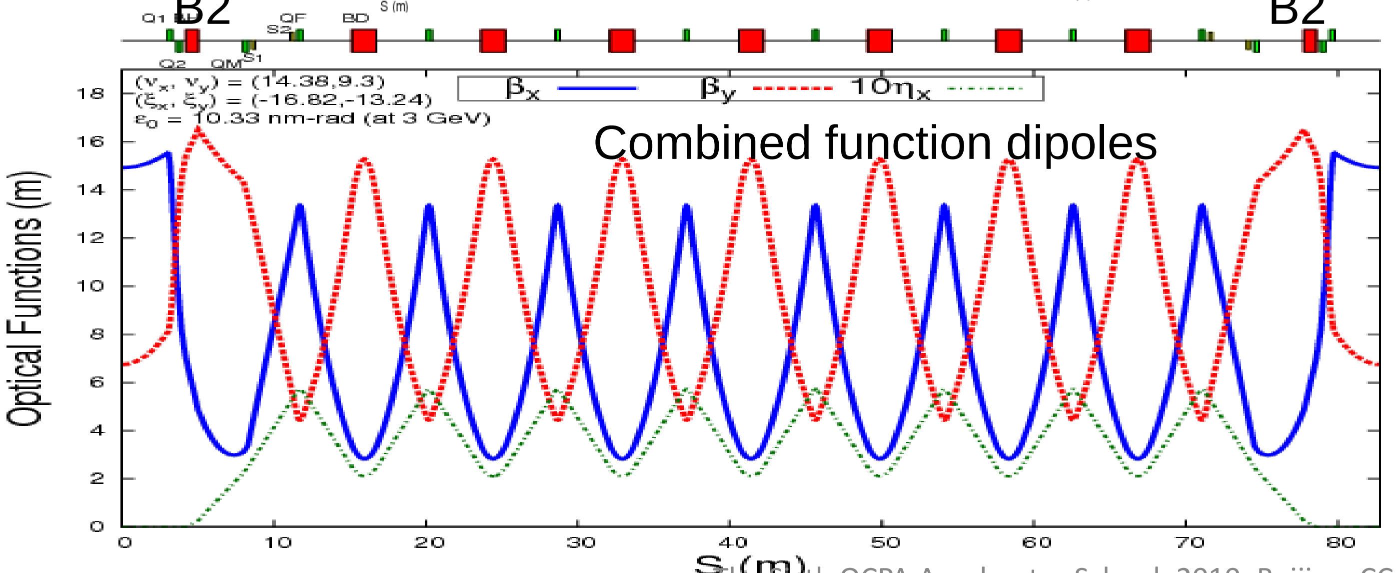
B1=8.823°
B2=6.176°





B1=8.57°
B2=4.28°

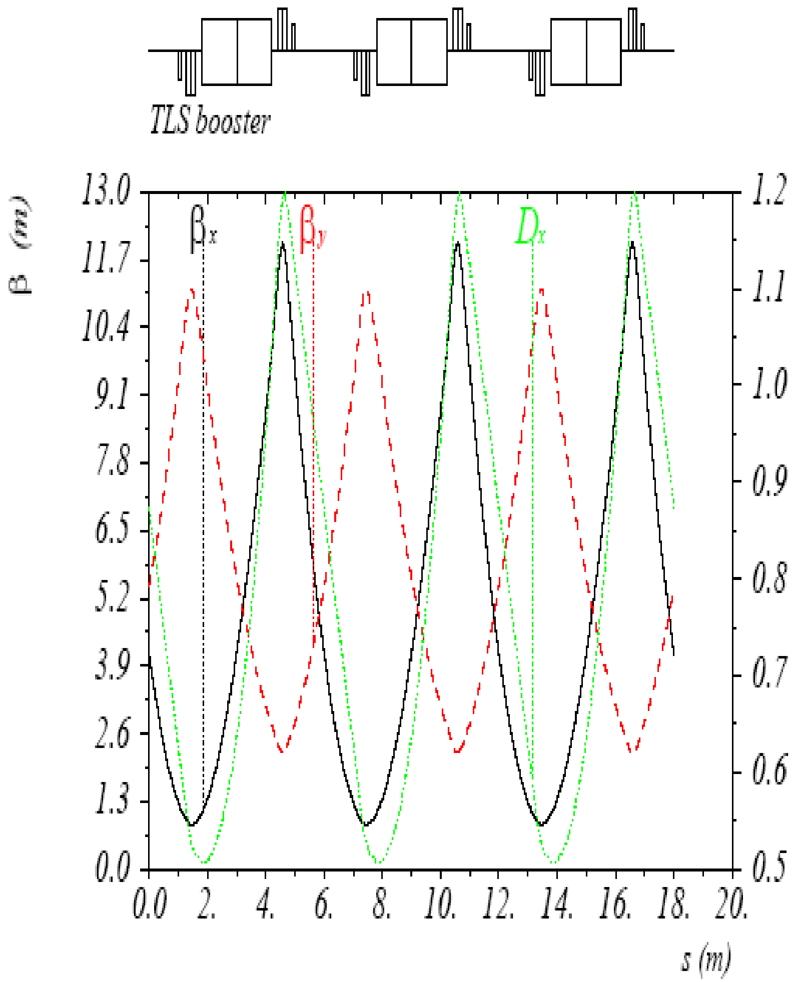
B2
B2

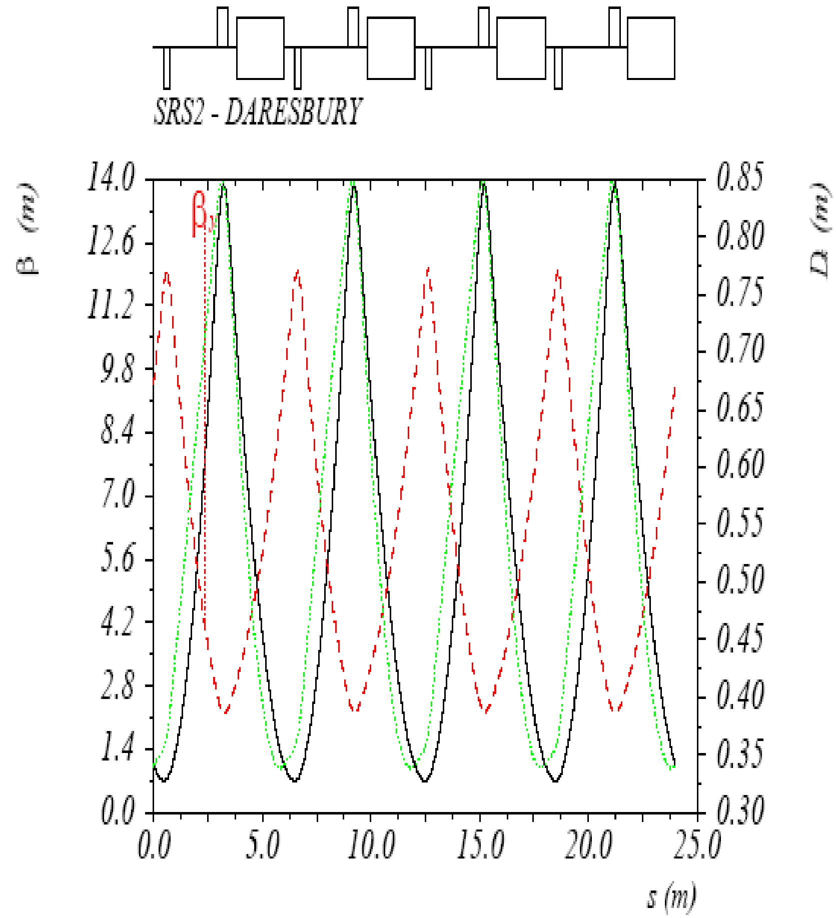


Combined function dipoles

B1=7.5°
B2=3.75°

FODO Lattice





Dispersion suppression in FODO type structure

- For FODO cells structure, dispersion suppression section at both ends of the standard cells.
- Usually, the solution can be with one bend (same or smaller angle) and proper phase advance. It can be with two bends with smaller angle. The figure shown below is the case with two bends and same quadrupole strengths and relations are as :

$$\theta_1 = \theta\left(1 - \frac{1}{4\sin^2\phi}\right)$$

$$\theta_2 = \theta\left(\frac{1}{4\sin^2\phi}\right)$$

$$\theta = \theta_1 + \theta_2$$

θ and ϕ : bend angle and phase advance in normal FODO cell

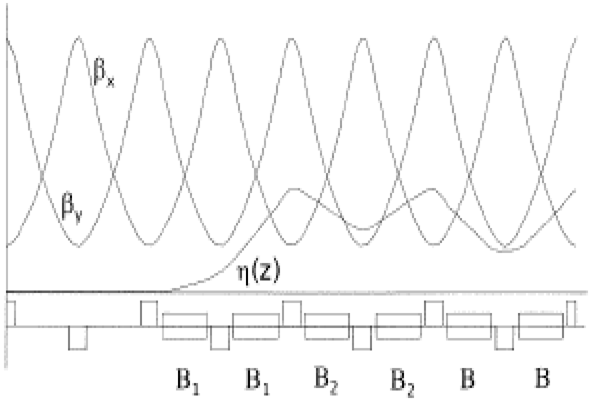
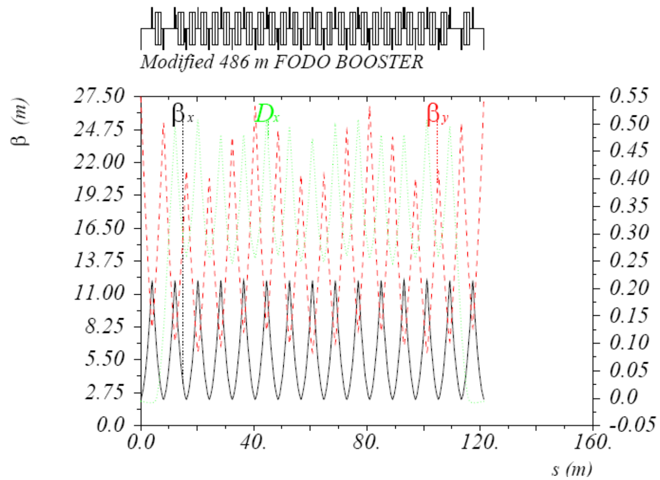
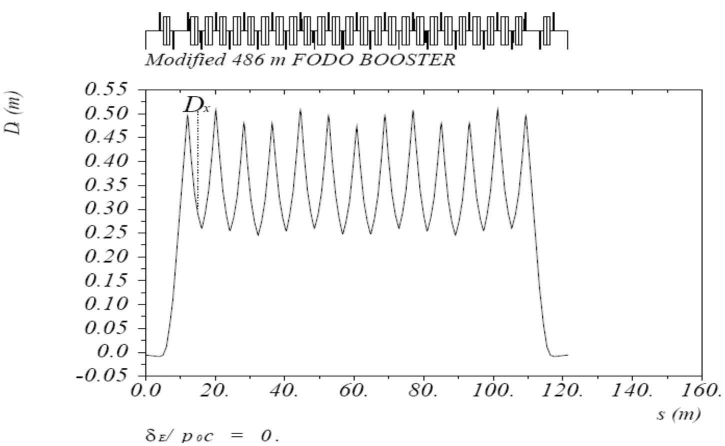


Fig. 7.14. Dispersion suppressor lattice

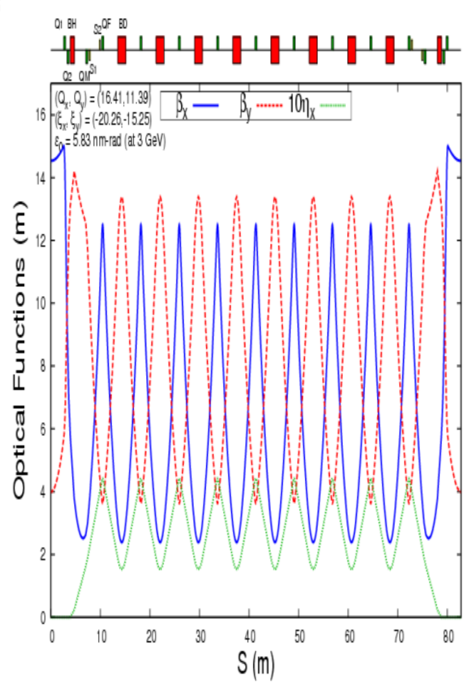
Dispersion suppression in FODO type structure (so called nBA?)

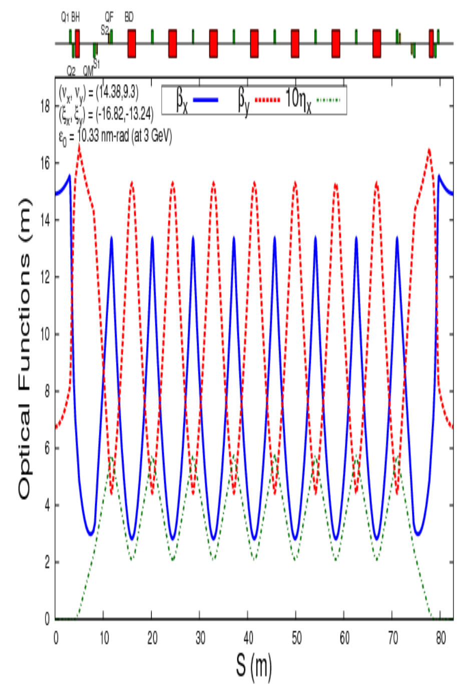
Same dipole
Separated functions dipole
Some small beta-beat





One smaller dipole
Combined-function dipole (defocusing)





Momentum compaction

$$\Delta C = \oint \frac{D}{\rho} ds\, \delta,\ \ \alpha_c = \frac{\Delta C}{C}\frac{1}{\delta} = \frac{1}{C}\oint \frac{D}{\rho} ds \cong \frac{1}{C}\sum_i \langle D_i \rangle \theta_i$$

$$\text{FODO cell}:\ \alpha_{c,FODO} \approx \frac{(D_F + D_D)\theta}{2L} \approx \frac{\theta^2}{\sin^2(\Phi/2)} \approx \frac{1}{\nu_x^2}$$

$$\text{DBA cell}:\ \alpha_{c,DBA} \approx \frac{\rho\theta^2}{6R},\ \text{R}:\text{average radius of ring}$$

For small bend angle and large circumference DBA lattice, the first-order momentum compactor can be in an order of 10^{-4}. Higher-order momentum compaction play important role in the longitudinal motion if the first-order is small.

Chromaticity Effects

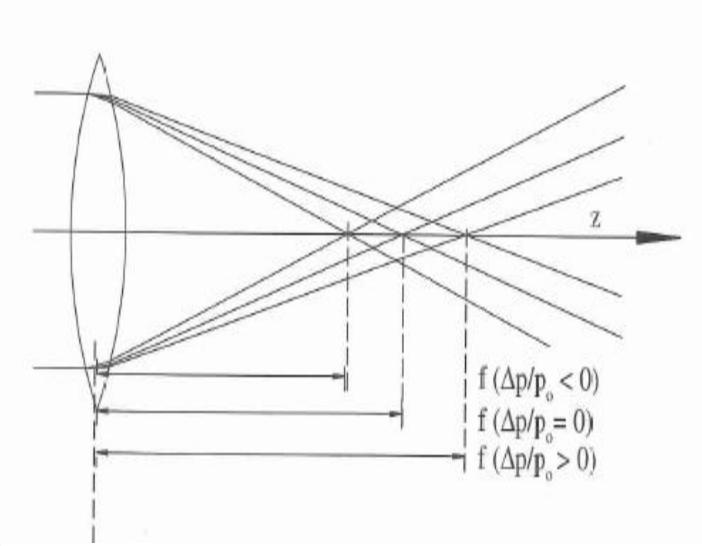
For off-momentum particle, gradient errors:

$$\Delta K_x = [-\frac{2}{\rho^2} + K(s)]\delta + O(\delta^2) \approx [-\frac{1}{\rho^2} - K_x(s)]\delta \approx -K_x(s)\delta,$$

$$\Delta K_z = -K(s)\delta + O(\delta^2) \approx -K_z(s)\delta$$

$$\Delta \nu_x = \frac{1}{4\pi}\oint \beta_x \, \Delta K_x ds \approx \frac{-1}{4\pi}\oint \beta_x \, K_x ds \, \delta$$

$$\Delta \nu_z = \frac{1}{4\pi}\oint \beta_z \, \Delta K_z ds \approx \frac{-1}{4\pi}\oint \beta_z \, K_z ds \, \delta$$



Chromaticity :

$$\xi = \frac{d(\Delta\nu)}{d\delta}, \text{natural chromaticity } \xi_{nat} = \frac{-1}{4\pi}\oint \beta \, K ds$$

FODO N cells :

$$\xi_{nat}^{FODO} = \frac{-1}{4\pi} N(\beta_{\max} - \beta_{\min})/f = -\frac{\tan(\Phi/2)}{\Phi/2}\nu \approx -\nu$$

Chromaticity correction

For a large (negative) natural chromaticity, the tune shift is large with typical energy offset and can cause beam storage lifetime reduction induced by transverse resonances. We need sextupole magnets installed in the storage ring to increase the focusing strength for larger energy beam. Two types of sextupoles, i.e., focusing and defocusing sextupoles, are properly situated. To avoid head-tail instability, a slightly positive chromaticity is preferred.

$$\frac{\Delta B_z}{B_0\rho} = \frac{B_2}{2B_0\rho}(x^2 - z^2),\ \frac{\Delta B_x}{B_0\rho} = \frac{B_2}{B_0\rho}xz,\ B_2 = \partial^2 B_z/\partial x^2 |_{x=z=0}$$

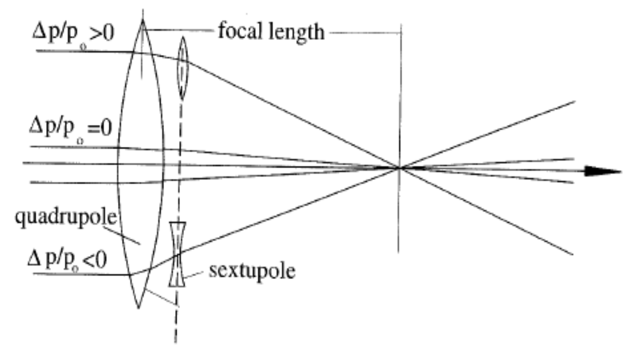
$$\text{Let } x = x_\beta + D\delta \Rightarrow \frac{\Delta B_z}{B_0\rho} = -[SD\delta]x_\beta - S(x_\beta^2 - z_\beta^2) - \frac{S}{2}D^2\delta^2$$

$$\text{and } \frac{\Delta B_x}{B_0\rho} = -[SD\delta]z_\beta - \frac{S}{2}x_\beta z_\beta, \text{ where } S = -\frac{B_2}{B_0\rho}$$

$$\text{To the first order } \Delta K_x = SD\delta,\ \Delta K_z = -SD\delta$$

$$\xi_x = \frac{-1}{4\pi}\oint \beta_x [K_x - S_F D_F - S_D D_D]ds$$

$$\xi_z = \frac{-1}{4\pi}\oint \beta_z [K_z + S_F D_F + S_D D_D]ds$$



$$\text{For FODO}: S_F\ell_F = \frac{1}{2f^2\theta}\frac{\sin\frac{\Phi}{2}}{(1+\frac{1}{2}\sin\frac{\Phi}{2})},\ S_D\ell_{SD} = -\frac{1}{2f^2\theta}\frac{\sin\frac{\Phi}{2}}{(1-\frac{1}{2}\sin\frac{\Phi}{2})}$$

Design Rules for sextupole scheme

- To minimize chromatic sextupoles strengths, it should be located near quadrupoles at least for FODO cells, where $\beta_x D_x$ and $\beta_z D_x$ are maximum.
- A large ratio of β_x/β_z for the focusing and β_y/β_z for the defousing sextupoles are needed.
- The families of sextupoles should be arranged to minimize resonance strengths.
- For strong focusing (low emittance) lattice, strong chromatic sextupole fields are needed to correct chromaticity and such strong nonlinear fields can induce strong nonlinear chromatic effects as well as geometric aberrations. Phase cancellations can help reduce nonlinear chromatic aberrations. Harmonic sextupoles are installed at proper positions to further reduce resonance strengths of geometric aberrations.

Nonlinear Effects of Chromatic Sextupoles

$$x'' + K_x(s)x = -S(s)(x^2 - z^2), \quad z'' + K_y(s)z = S(s)xz$$

$$x = x_\beta + D\delta$$

$$\text{normalized coordinates: } w_\beta = \frac{x}{\sqrt{\beta_x}}, \quad w_\delta = \tilde{D}\delta = \frac{D}{\sqrt{\beta_x}}\delta$$

$$\tilde{D}'' + \nu_0^2 \tilde{D} = \nu_0^2 \beta^{3/2} / \rho + \nu_0^2 \beta^2 k \tilde{D}\delta - \frac{1}{2}\nu_0^2 \beta^{5/2} S \tilde{D}^2 \delta,$$

$$w_\beta'' + \nu_0^2 w_\beta = \nu_0^2 \beta^2 k w_\beta \delta - \nu_0^2 \beta^2 S D w_\beta \delta - \frac{1}{2}\nu_0^2 \beta^{5/2} S w_\beta^2$$

- The first two terms cancel the chromatic aberrations to first order locally. However, in real machine, local cancellation might be not perfect and beta-beat and higher harmonics of chromatic terms in the circular machine still exist .
- The third term is betatron ampltude dependent perturbation and called geometric aberrations. *-I transformation* scheme is to put sextupoles in (2n+1) π phase advance apart in a periodic lattice and this scheme can compensate the effects.

Chromatic Aberrations

Chromatic gradient errors with sextupoles

$$\Delta K(s) = (k - SD)\delta$$

Beta − beat :

$$\frac{\Delta\beta(s)}{\beta} = -\frac{\delta}{2\sin\Phi_0}\oint \beta(t)(k(t) - S(t)D(t))\cos[2\nu_0(\pi + \phi(s) - \phi(t)]dt$$

If chromaticity is corrected locally, there is no induced beta-beat and no half integer resonances. However, in reality, we might need several families of sextupoles to get small beta-beat

Collider need to have local chromatic aberration correction with sextuples at interaction point to ensure small beta-beat at IP and keep beam size as small as possible (high luminosity).

-*I* transformation

$$\text{phase advance} = (2n+1)\pi, n = 0,1,2..$$

$$M = -I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

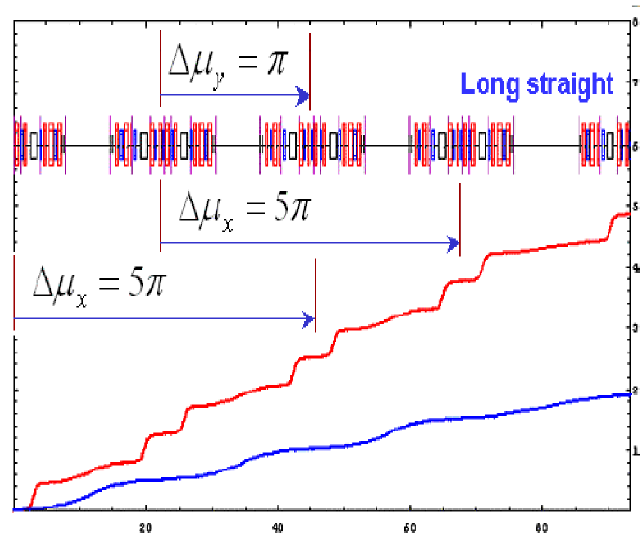
$$\Delta x' = -\frac{1}{2} S\ell(x^2 - y^2)$$

$$\Delta y' = -S\ell xy$$

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = M_s(x_0, y_0) \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} S\ell x_0 & 1 & -\frac{1}{2} S\ell y_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & S\ell x_0 & 1 \end{pmatrix}$$

$$M_t = M_s(x_0, y_0) \cdot (-I) \cdot M_s(-x_0, -y_0) = -I$$

For periodic lattice, compensation can across one or several cells. Thick-lens sextupoles together with errors in real machine such as COD, quad gradient errors, and working point selection always lead some limitation in its effectiveness.



Diamond phase cancellation scheme. Horizontal tune ~ 29.12 due to long straight perturbation

Selection of working tune is different from above. Diamond Light Source nominal lattice working tune (27.23, 12.36)

Nonlinear Hamiltonian

$$H=\frac{1}{2}(x'^2+K_x x^2+y'^2+K_y y^2)+\frac{S(s)}{2}(x^3-3xy^2)$$

$$H=H_0+V_3(x,y,s)$$

$$H_0=\frac{1}{2}(x'^2+K_x x^2+y'^2+K_y y^2),\ \ V_3=\frac{S(s)}{2}(x^3-3xy^2)$$

$$V_3=-\frac{\sqrt{2}}{4}J_x^{1/2}J_y\beta_x^{1/2}\beta_y S(s)[2\cos\Phi_x+\cos(\Phi_x+2\Phi_y)+\cos(\Phi_x-2\Phi_y)]$$

$$+\frac{\sqrt{2}}{12}J_x^{3/2}\beta_x^{3/2}S(s)[\cos 3\Phi_x+3\cos\Phi_x]$$

$$\Phi_x=\phi_x+\chi_x(s)-\nu_x\theta,\ \ \chi_x(s)=\int_0^s\frac{ds}{\beta_x}$$

$$\Phi_y=\phi_y+\chi_y(s)-\nu_y\theta,\ \ \chi_y(s)=\int_0^s\frac{ds}{\beta_y}$$

$(J_x,\phi_x),(J_y,\phi_y)$ are pairs of conjugate phase-space

Periodic in s, Fourier expansion :

$$H=\nu_x J_x+\nu_y J_y+\sum_{\ell}G_{3,0,\ell}J_x^{3/2}\cos(3\phi_x-\ell\theta+\zeta_{3,0,\ell})$$

$$+\ldots J_x^{1/2}J_y\cos(\phi_x+2\phi_y-\ell\theta+\zeta_{1,2,\ell})$$

$$+\ldots J_x^{1/2}J_y\cos(\phi_x-2\phi_y-\ell\theta+\zeta_{1,-2,\ell})$$

$$+\ldots$$

$$G_{3,0,\ell}e^{j\zeta}=\frac{\sqrt{2}}{24\pi}\oint\beta_x^{3/2}S(s)e^{j[3\chi_x(s)-(3\nu_x-\ell)\theta]}ds$$

Nonlinear Effects of Chromatic Sextupoles

$$\text{Sum resonance}: v_x + 2v_y = \ell$$

$$\text{Difference resonance}: v_x - 2v_y = \ell$$

$$\text{Parametric resonance}: v_x = \ell,\ 3v_x = \ell$$

$$\text{other higher-order resonances}: 4v_x = \ell, 2v_x \pm 2v_y = \ell \ldots$$

Concatenation of sextupoles perturbation to the betatron motion can induce nonlinear betatron detuning.

$$v_x = v_{x0} + \alpha_{xx} J_x + \alpha_{xy} J_y$$

$$v_y = v_{y0} + \alpha_{xy} J_x + \alpha_{yy} J_y$$

$$\alpha_{xx}, \alpha_{xy}, \alpha_{yy} : \text{detuning coefficient}$$

Sextupole Hamiltonian

For the first 9 terms of first-order driving sources: 4 Chromatic terms ,5 Geometry terms→ 9 families can get first-order optimal solutions

$$h_{jklmp} \propto \sum_{n}^{Nsxt} (b_3L)_n \beta_{xn}^{\frac{j+k}{2}} \beta_{yn}^{\frac{l+m}{2}} D_n^p e^{i[(j-k)\varphi_{xn}+(l-m)\varphi_{yn}]} - [\sum_{n}^{Nquad} (b_2L)_n \beta_{xn}^{\frac{j+k}{2}} \beta_{yn}^{\frac{l+m}{2}} e^{i[(j-k)\varphi_{xn}+(l-m)\varphi_{yn}]}]_{p\neq 0}$$

For the first 12 terms: 4 Chromatic terms ,5 Geometry terms,3 Amplitude dependent terms (second-order terms and critical) need to be minimized.

$$\begin{aligned} f &= p_1 |h_{21000}(\nu_x)|^2 + p_2 |h_{10110}(\nu_x)|^2 \\ &+ p_3 |h_{30000}(3\nu_x)|^2 \\ &+ p_4 |h_{10020}(\nu_x + 2\nu_y)|^2 + p_5 |h_{10200}(\nu_x - 2\nu_y)|^2 \\ &+ p_6 |d\nu_x / dJ_x|^2 + p_7 |d\nu_x / dJ_y|^2 \\ &+ p_8 |d\nu_y / dJ_y|^2 \\ &+ p_9 |h_{11001}(\xi_x)|^2 + p_{10} |h_{00111}(\xi_y)|^2 \\ &+ p_{11} |h_{20001}(2\nu_x(\delta))|^2 + p_{12} |h_{00201}(2\nu_y(\delta))|^2 \end{aligned}$$

Object Function Including Higher-order Driving terms

Usually, we can get good solution for the first 12 terms optimization. However, we can further reduce 2^{nd}-order terms including 11 more terms.

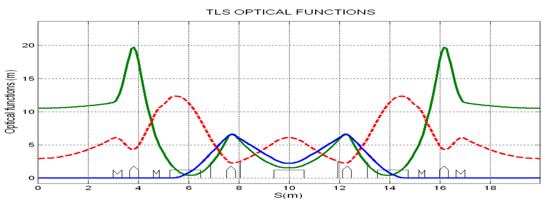
13 terms in 2nd -order of sextupole strength

- 3 linear tune shift with amplitude
- 8 octupole-like driving terms $4\nu_x, 2\nu_x \pm 2\nu_y$
- 2 terms generating second-order chromaticity

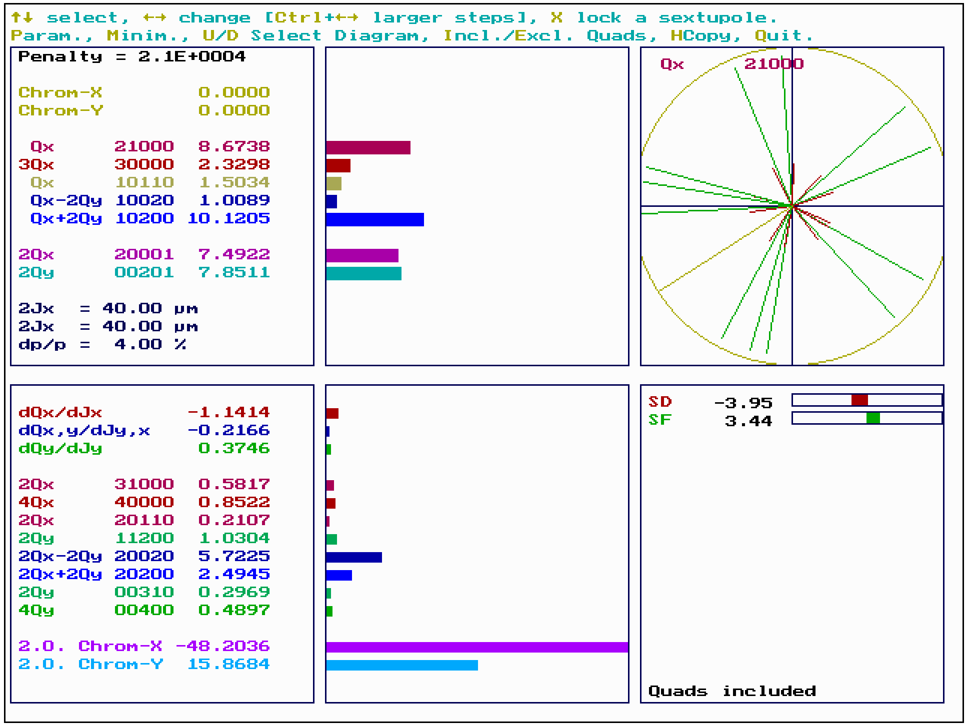
This nonlinear optimization is very important for the low emittance, high chromaticity lattice. Iterations between linear and nonlinear schemes are proceeded to get acceptable solution.

TLS Storage Ring

For TLS, do we need more than two sextupole families?

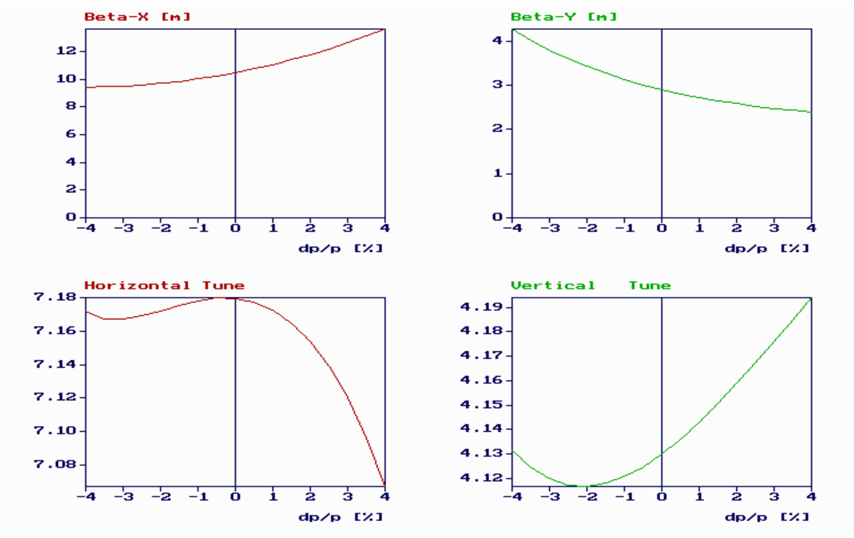


Lattice type	**TBA**
Operational energy	**1.5 GeV**
Circumference	**120 m**
Natural emittance	*25.6 nm-rad (achromat)*
Natural energy spread	*0.075%*
Momentum compaction factor	*0.00678*
Damping time	
Horizontal	*6.959 ms*
Vertical	*9.372 ms*
Longitudinal	*5.668 ms*
Betatron tunes horizontal/vertical	*7.18/4.13*
Natural chromaticities	
Horizontal	***-15.292***
Vertical	***-7.868***
Radiation loss per turn (dipole)	128 keV



Only two families of sextupoles for chromaticity correction.

TLS Storage Ring



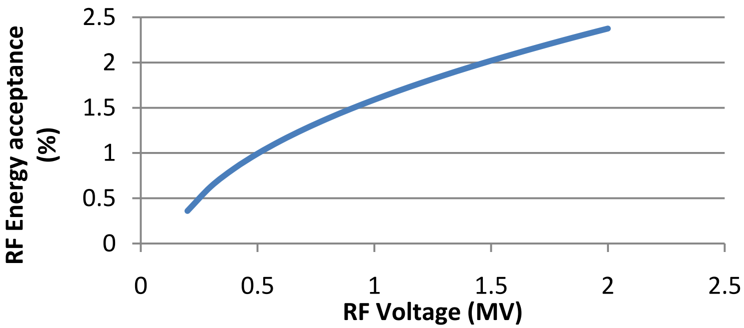
Beta and tune change vs energy

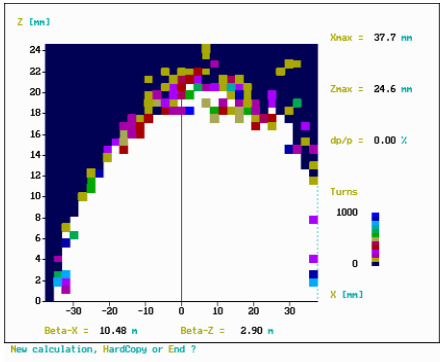
What is the RF energy acceptance?

$$\left(\frac{\Delta p}{p_0}\right)^2_{acc} = \frac{eV_0 \sin\psi_s}{\pi h \alpha_c c p_0}\left(\sqrt{q-1} - \cos^{-1}(1/q)\right)$$

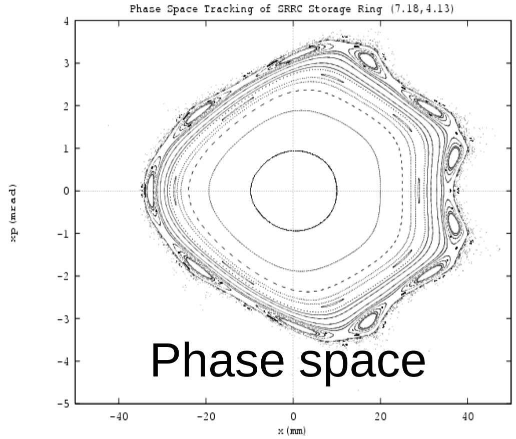
$$q = \frac{eV_0}{U_0} = \frac{1}{\sin\psi_s}$$

$$\text{TLS}: U_0 = 128 keV, \text{ No ID}$$

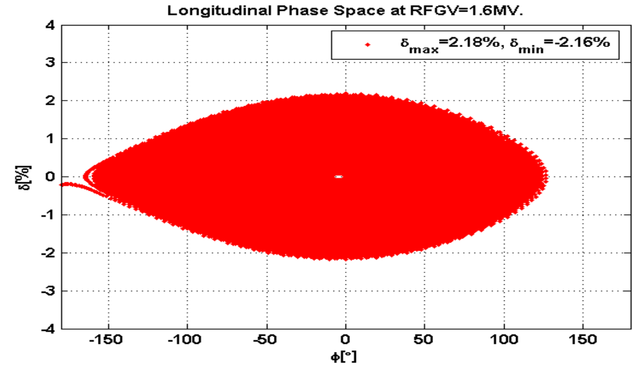




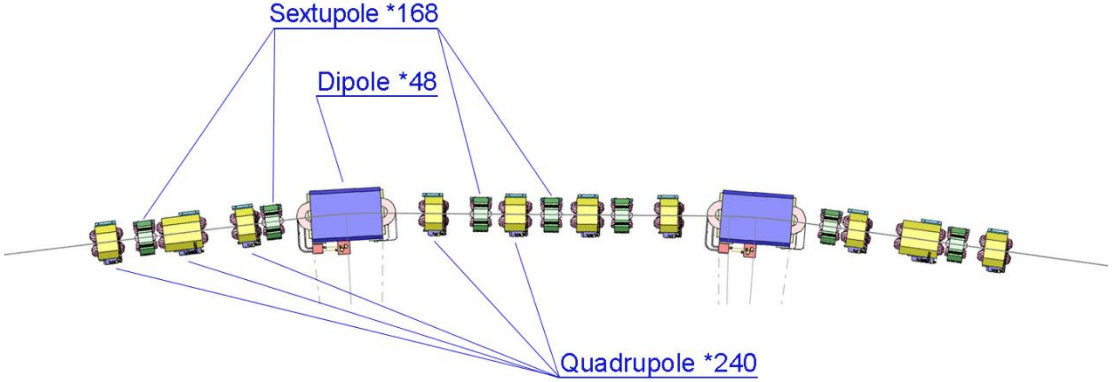
Dynamic aperture

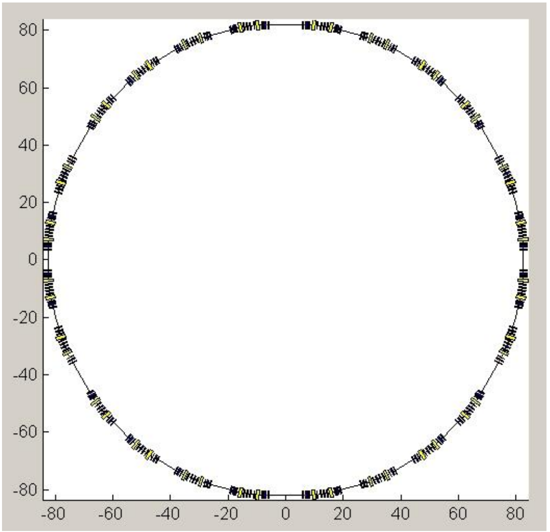


Phase space



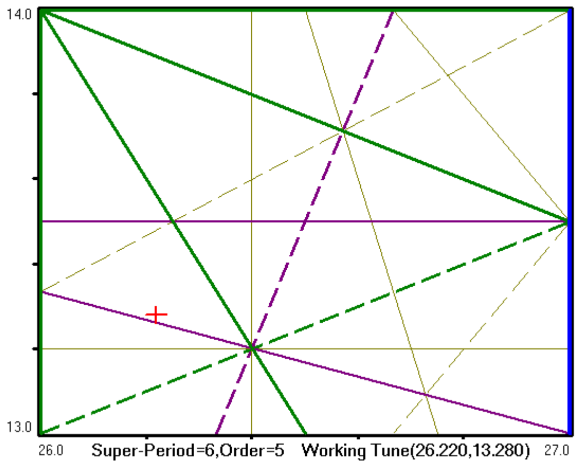
TPS Lattice Structure





Betatron Tune and Nonlinear Resonance

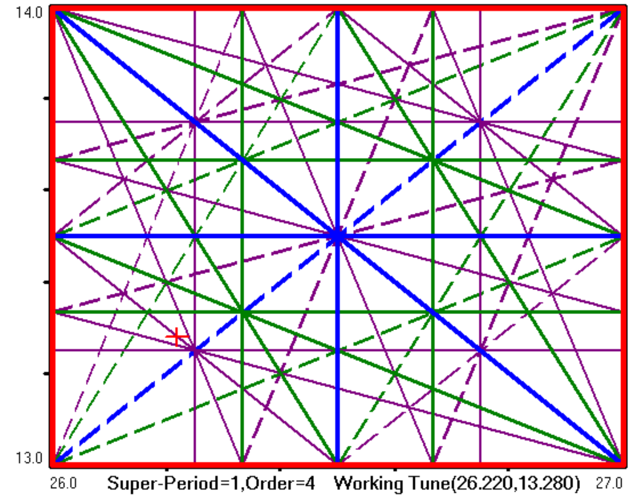
Tune selection in the lattice very important. Tune should be away from integer, half integer and third-order resonances. Introduction of sextupoles and nonlinear field errors in the magnets can drive higher-order nonlinear resonances. Particle tracking study and fequency (tune)-map analyses can further provide optimization information in tune selection.



Systematic resonances

$$n\nu_x \pm m\nu_y = 6p$$

n, m, p are integers



Random resonances

$$n\nu_x \pm m\nu_y = p$$

n, m, p are integers

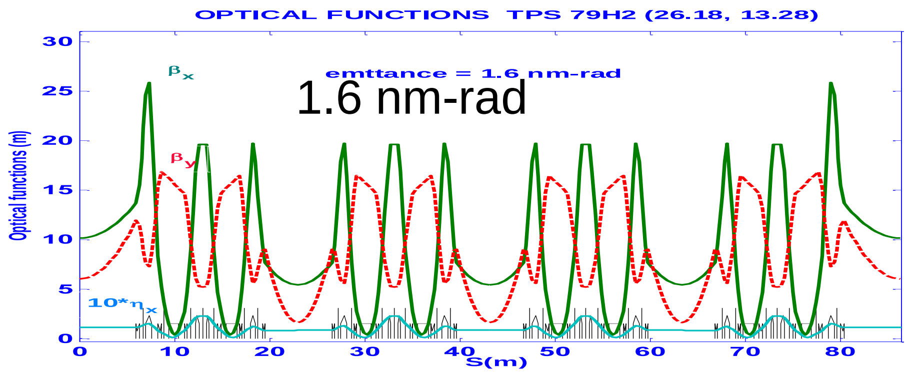
Matching Conditions

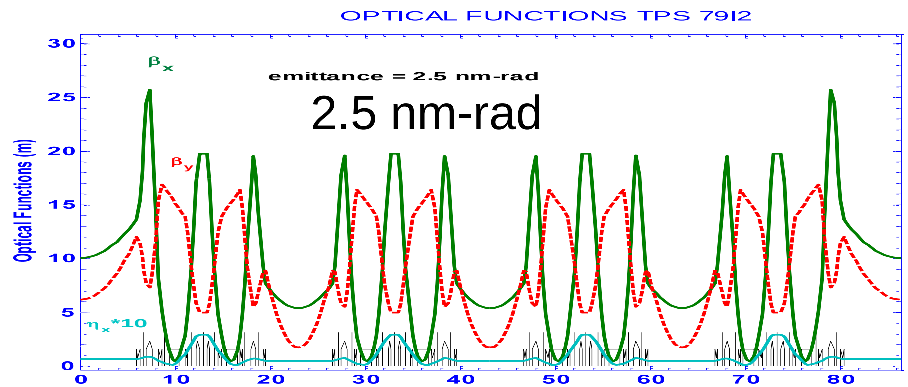
- **long straight sections ~ 12 m (quad to quade)**
- **standard straight sections ~ 7 m (quad to quad)**
- **circumference 518.4 m, 24 cells, 6-fold**
- **β_x ~10 m in long straight sections for injection**
- **β_x ~ 5 m in standard straight sections for ID**
- **β_y < 3.5 m in standard straight sections for ID**
- **reasonable β_x and β_y < 30 m anywhere in the whole ring**
- **fractional horizontal tune < 0.5**
- **fractional vertical tune < 0.5**
- **working point away from systematic and random resonances**
- **large betatron decoupling in the arc for sextupole chromaticity correction**
- **large dispersion in the arc center to decrease the sextupole strength**

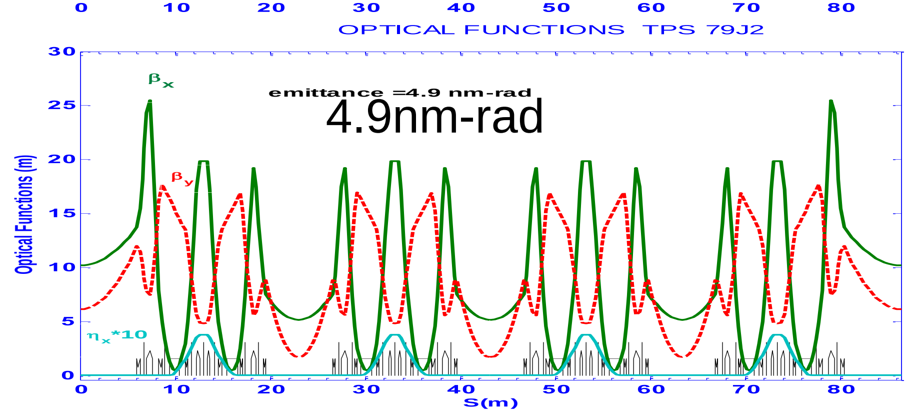
Linear Lattice Matching

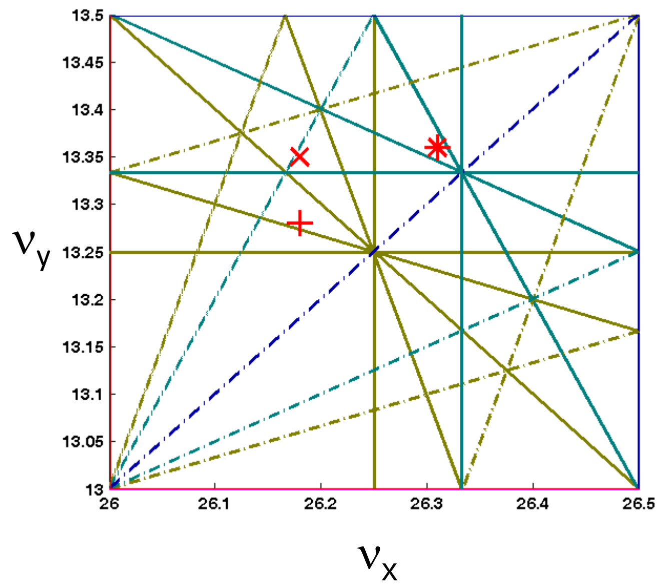
- Computer design codes are used in the matching. (for example MAD)
- Matching method such as simplex command is to minimize the penalty function by simplex method. Make sure enough varying parameters for the selected constraints.
- Starting from unit cell and impose constraints on optical functions such as D, D' at both ends of dipoles, local and global betatron functions, phase advance per cell, etc. Weighting factors are also given.
- Construct superperiod structure and do the same matching process with different constraints.
- Maximum strengths of quads are limited.
- Ring tunes are matched.
- Not always can find stable solutions and need change initial conditions for matching.
- Examine the global parameters and fitted parameters. If satisfactory, go to nonlinear optimization.

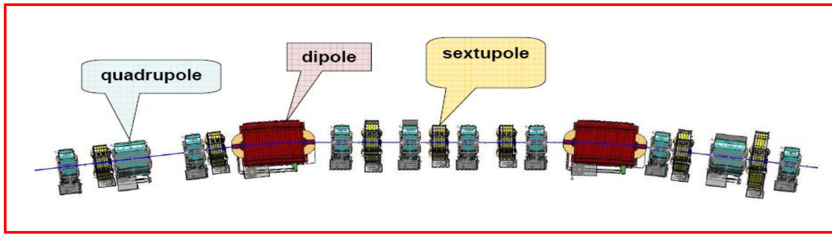
TPS Storage Ring Lattice Functions











24 DBA cells

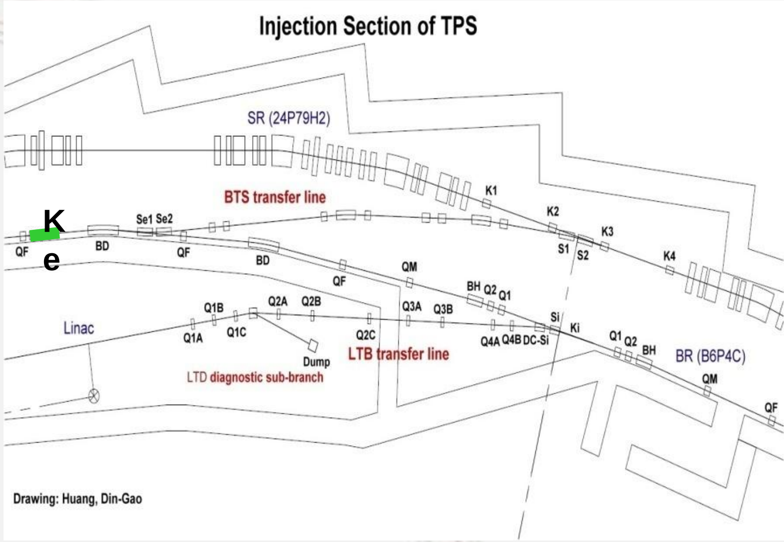
Major Parameters of TPS Storage Ring

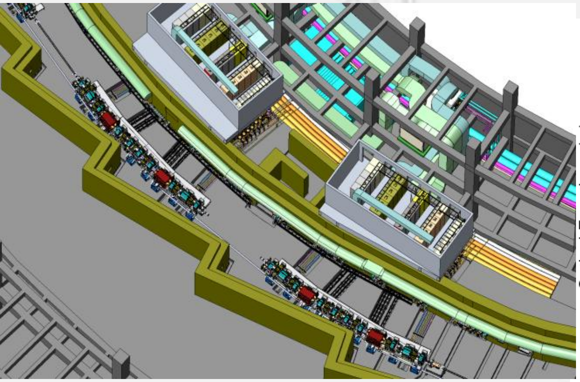
Circumference C (m)	518.4
Energy E (GeV)	3.0
Beam current (mA)	400
Natural emittance ε_{x0} (nm-rad)	1.6
Straight sections (m)	12 (x6) + 7 (x18)
Radiofrequency (MHz)	499.654
Harmonic number h	864
RF voltage (MV)	3.5
Energy loss per turn (dipole) (MeV)	0.85269
Betatron tune ν_x/ν_y	26.18 /13.28
Momentum compaction (α_1, α_2)	2.4×10^{-4}, 2.1×10^{-3}
Natural energy spread σ_E	8.86×10^{-4}
Damping time $\tau_x/\tau_y/\tau_s$ (ms)	12.20 / 12.17 / 6.08
Natural chromaticity ξ_x/ξ_y	-75 / -26
Synchrotron tune ν_s	0.00609
Bunch length (mm) σ_l	2.86

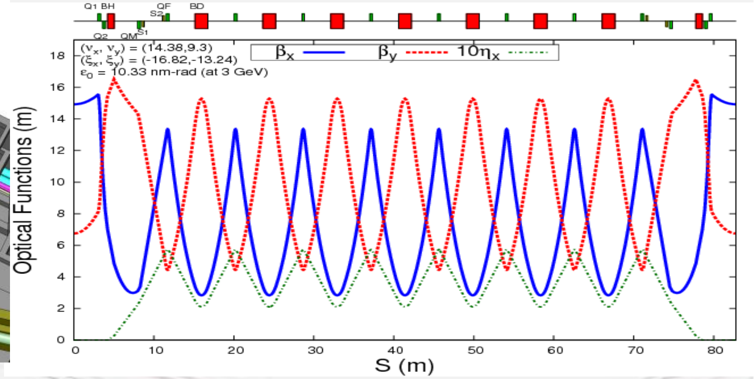
Source point	σ_x (μm)	$\sigma_{x'}$ (μrad)	σ_y (μm)	$\sigma_{y'}$ (μrad)
Long straight center	165.10	12.49	9.85	1.63
Short straight center	120.81	17.26	5.11	3.14
Dipole (1 degree source point)	39.73	76.11	15.81	1.11

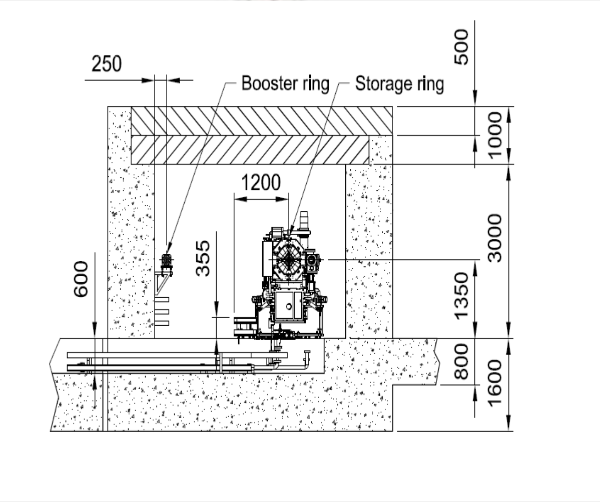
TPS injector parameters

Linac energy : 150 MeV
Booster energy: 3 GeV
Booster circumference: 496.8 m
Lattice structure: Modified FODO
Booster emittance: 10 nm-rad
Repetition rate: 3 Hz
RF: 500 MHz









Nonlinear effects

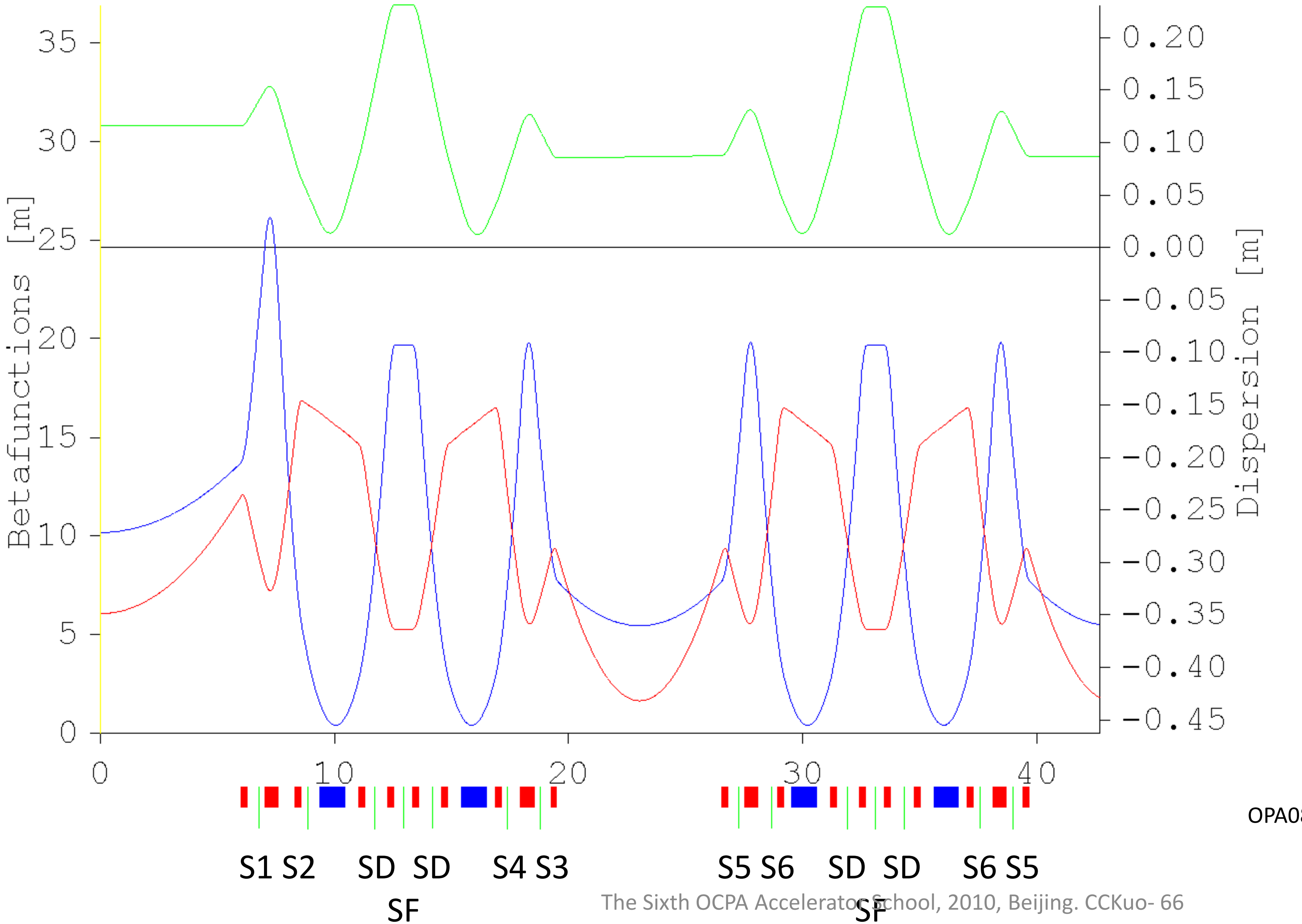
Sufficient dynamic aperture (maximum allowable particle oscillation amplitude) are necessary to guarantee long enough beam current lifetime and beam injection efficiency. So, we need to correct nonlinear effects to get acceptable dynamic aperture, both on and off-energy.

Low emittance
→ strong focusing quadrupoles
→ large chromatic aberration effects
→ strong sextupoles to correct chromaticities
→ strong nonlinear effects
→ sextupole scheme & nonlinear optimization
→ good dynamic aperture and energy acceptance (done)
→go back to linear lattice matching if not satisfied

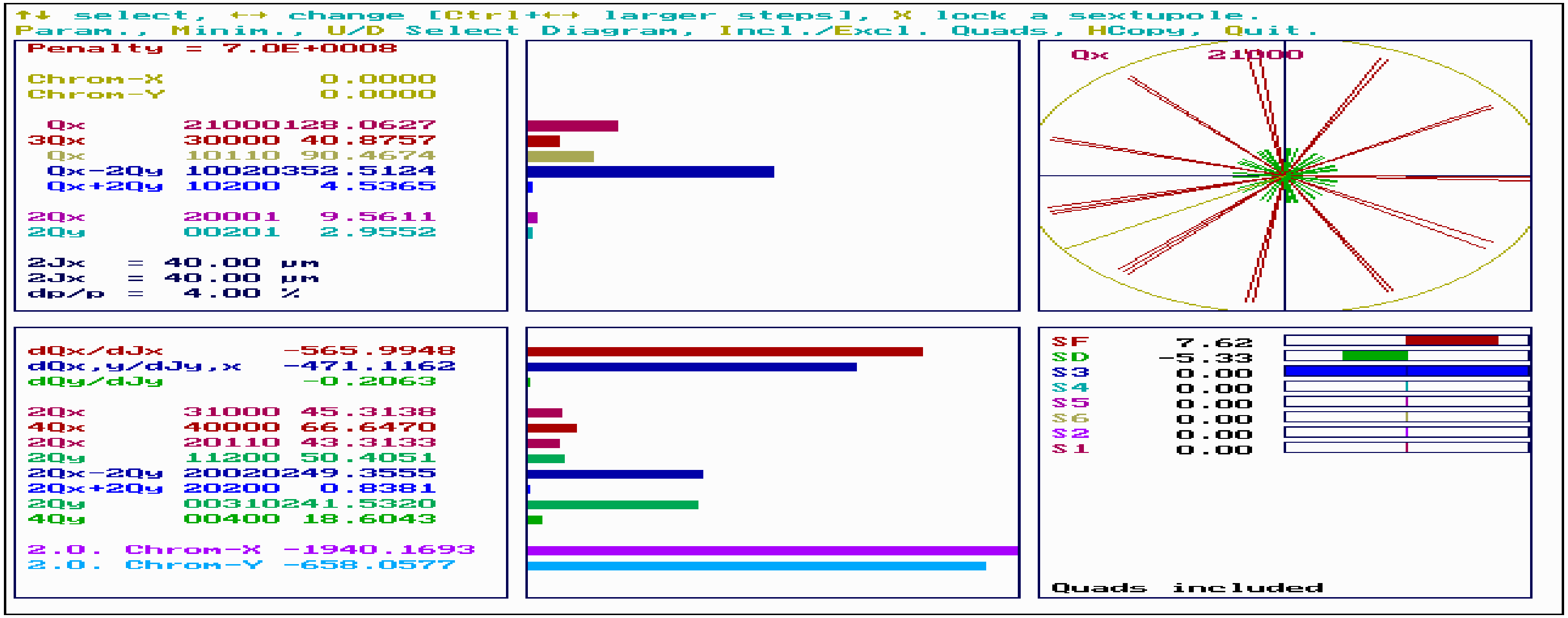
Nonlinear optimization

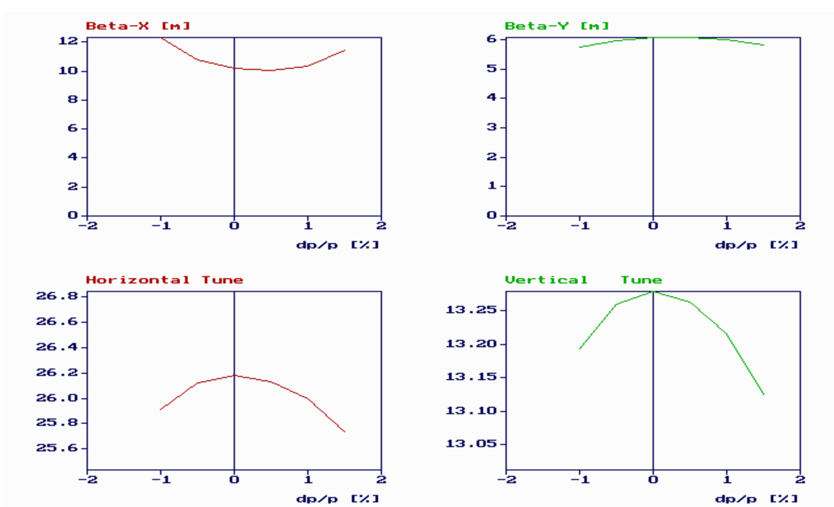
- **Quality factor: Tune shift with amplitude, Phase space plots,Tune shift with energy, Dynamic aperture (on and off momentum, 4D, 6D), Frequency Map Analysis**
- **Codes: OPA, BETA, Tracy-2, MAD, Patricia, AT, elegant, etc.**
- **8 families of sextupoles are used.**
- **Chromaticities are corrected to slightly positive.**
- **Weighting factors, sextupole families, positions are varied.**
- **Effects on the dynamic aperture in the presence of ID, field errors, chamber limitation, alignment errors, etc. are studied.**

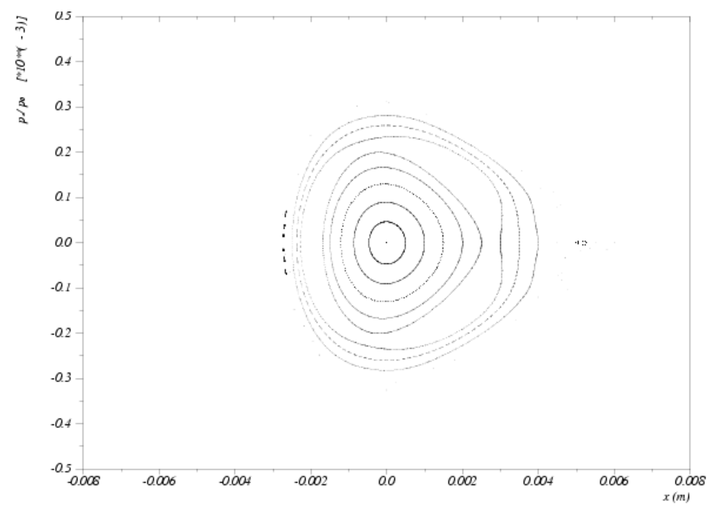
TPS Sextupole scheme



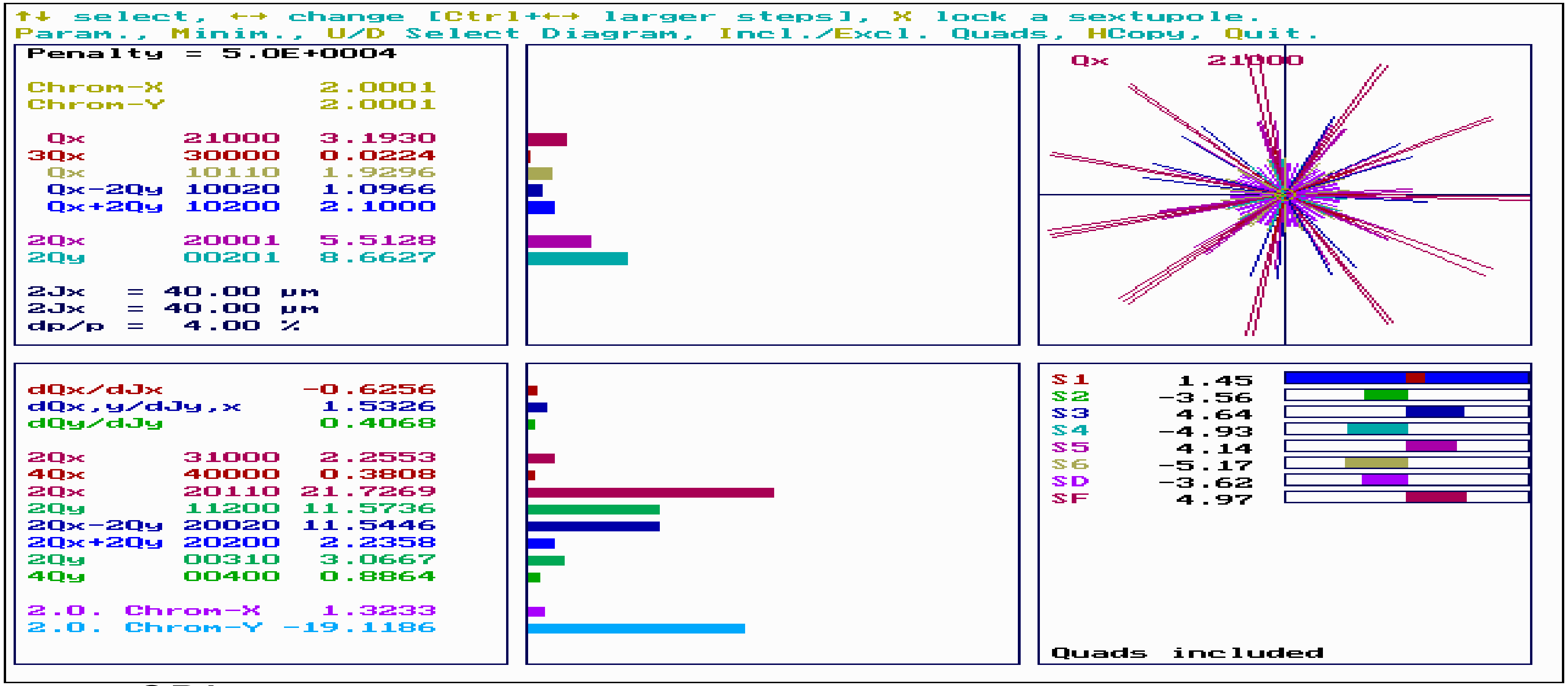
Only Chromatic Sextupoles







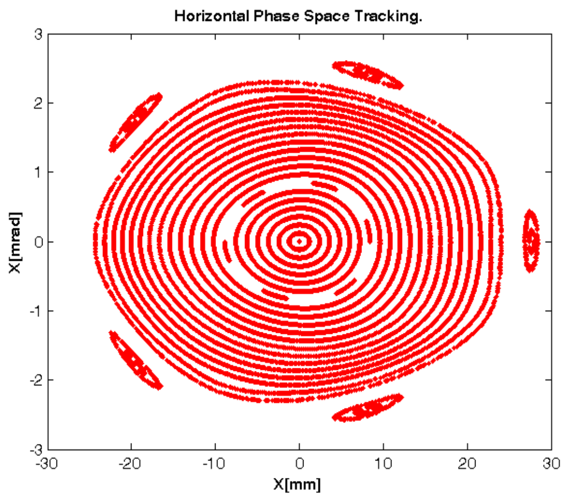
Nonlinear Optimization with Sextupoles

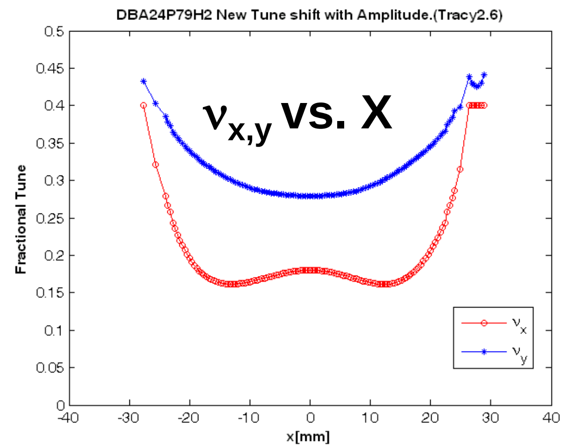


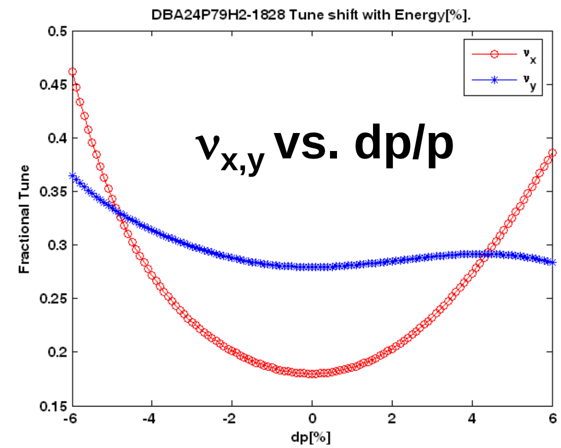
OPA

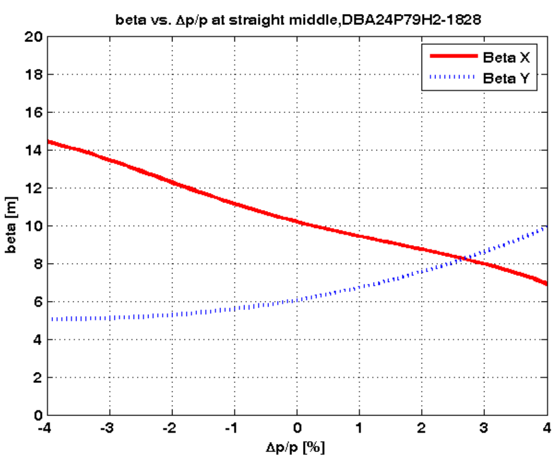
8 families of sextupoles for nonlinear optimization. Chromaticity of +5 in both planes are still with acceptable dynamic aperture and energy acceptance.

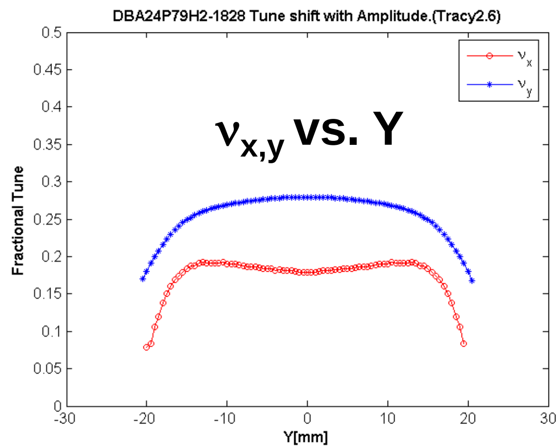
Phase Space
Betatron Function vs Energy
Tune Shifts vs. Amplitude and Energy
Dynamic Aperture

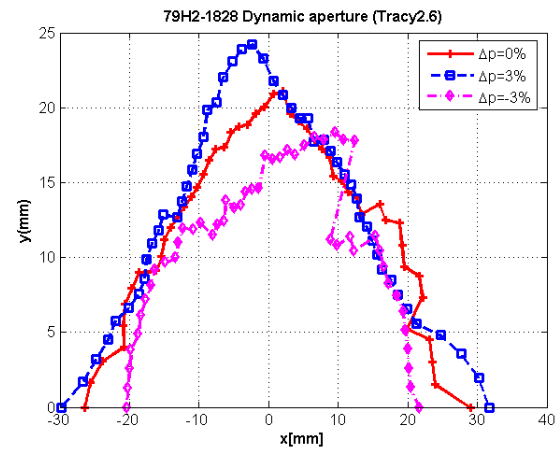




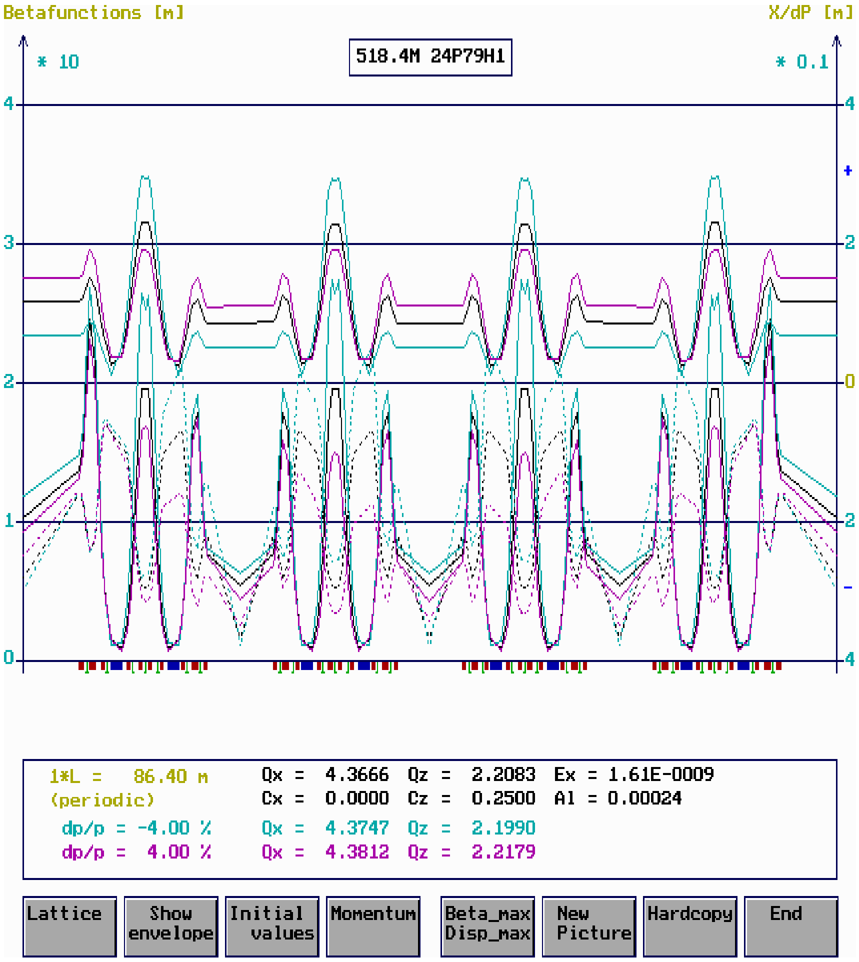


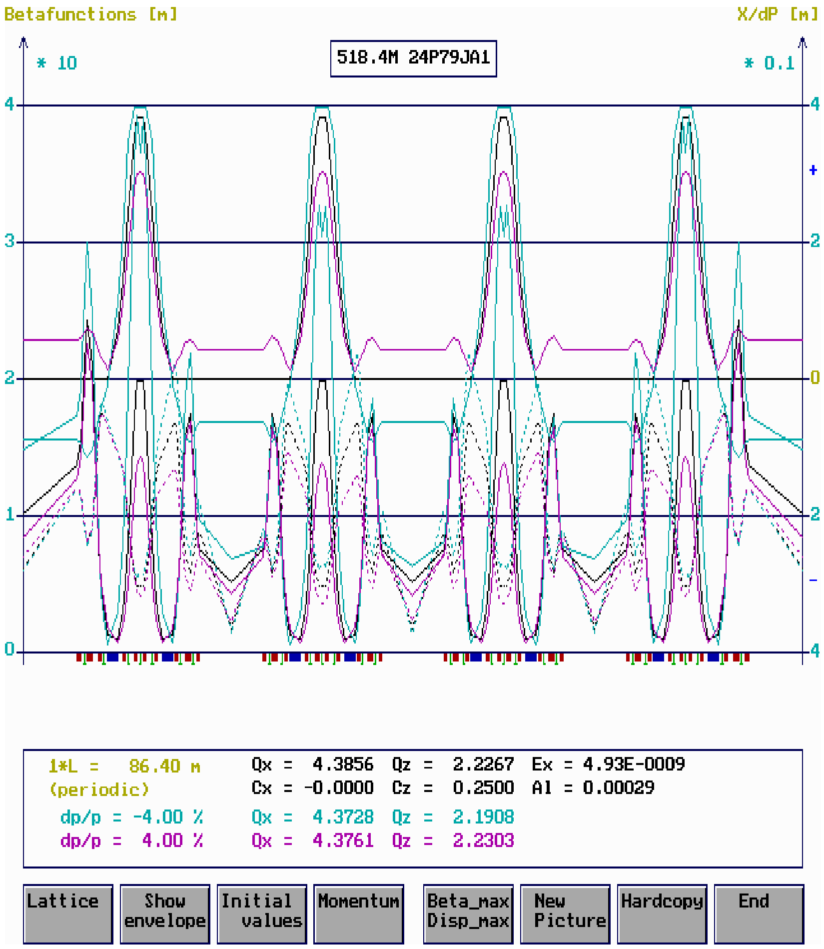






Off-momentum Optics Change





RF Energy Aperture
3.5 MV RF
(no chamber limitation)

79H2

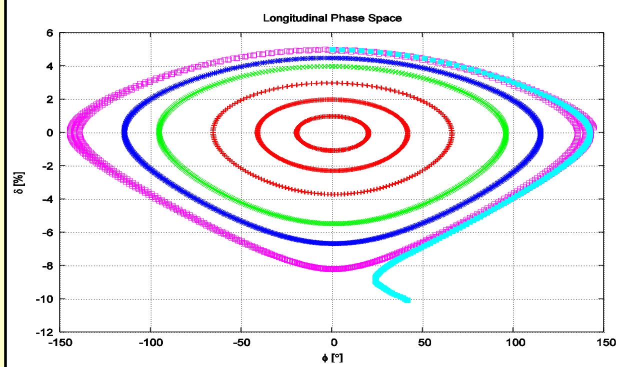
Momentum compaction factors

$$\alpha_1 = 2.4\times10^{-4}, \alpha_2 = 2.1\times10^{-3}$$

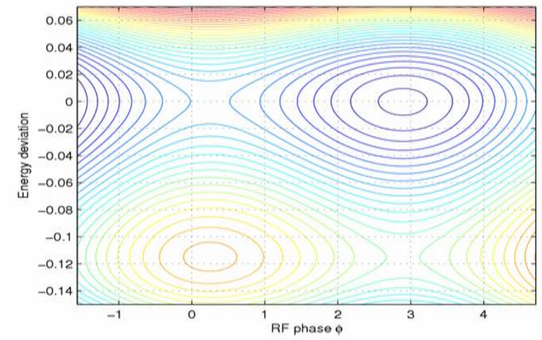
$$\alpha_1 = \frac{1}{L_0}\oint \frac{D_0}{\rho} ds$$

$$\alpha_2 = \frac{1}{L_0}\oint \left[\frac{D_0'^{\,2}}{2} + \frac{D_1}{\rho}\right] ds$$

$$\alpha = \alpha_1 + 2\alpha_2 \Delta p / p$$



Tracy-2

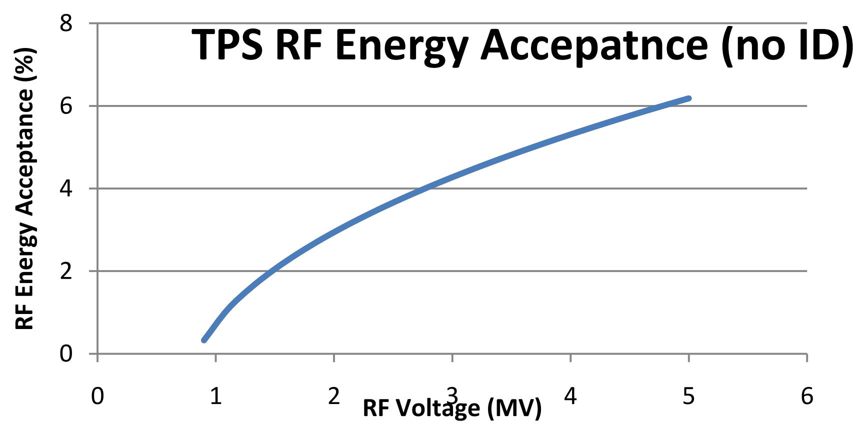


analytical

$$\left(\frac{\Delta p}{p_0}\right)^2_{acc} = \frac{eV_0 \sin\psi_s}{\pi h \alpha_c c p_0}\left(\sqrt{q-1} - \cos^{-1}(1/q)\right)$$

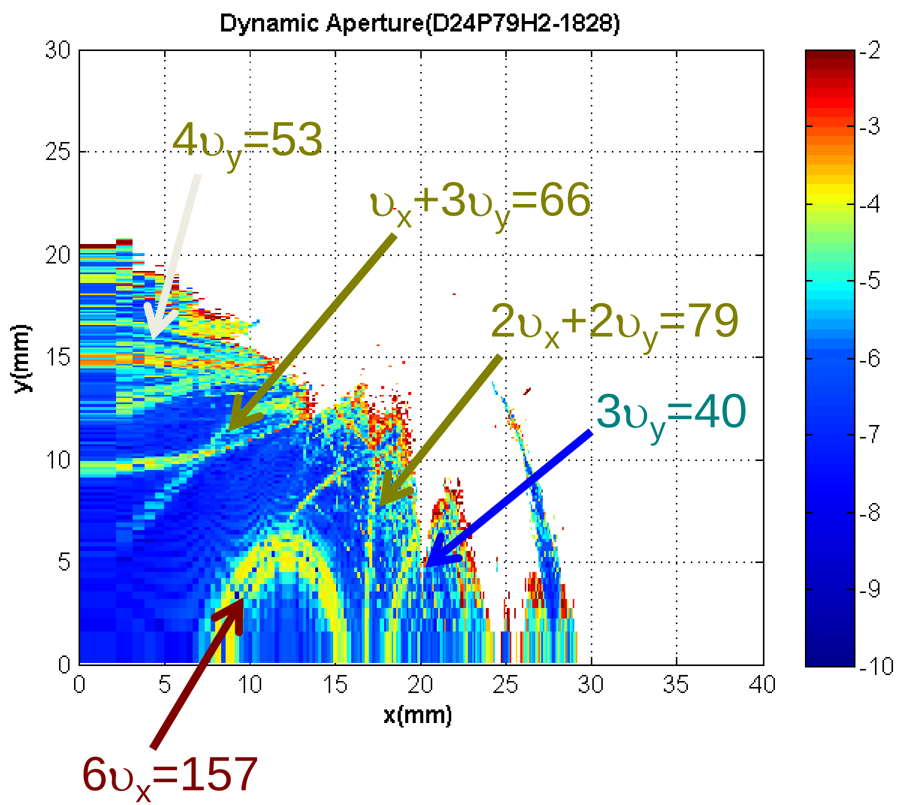
$$q = \frac{eV_0}{U_0} = \frac{1}{\sin\psi_s}$$

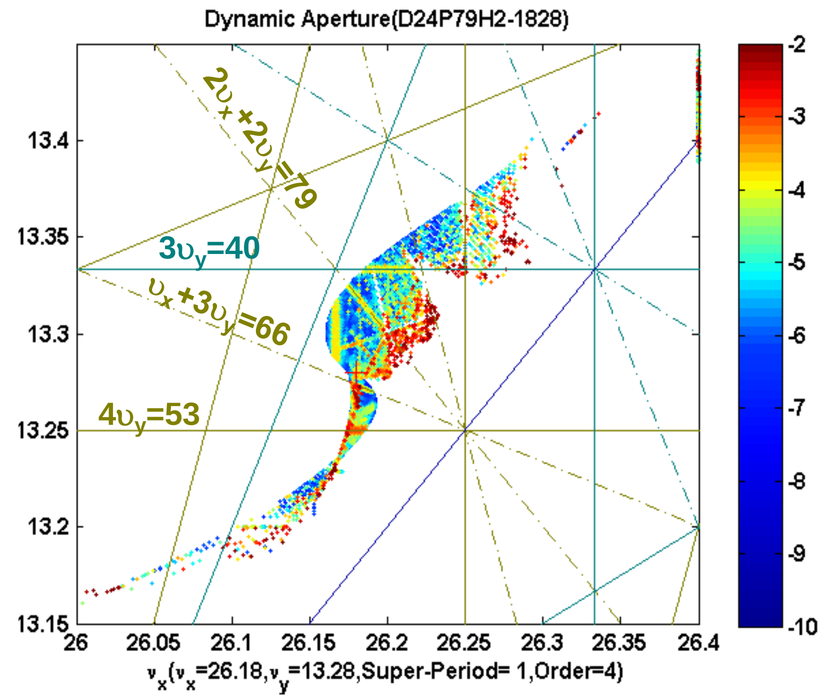
$$\text{TPS}: U_0 = 0.85 MeV, \text{No ID}$$



Dynamic Aperture and FMA

(26.18/13.28)

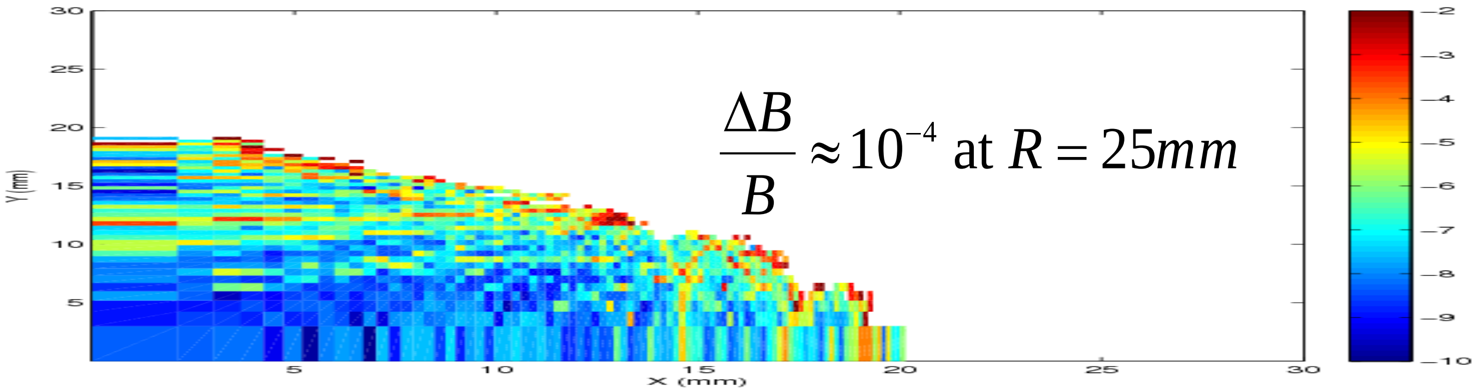


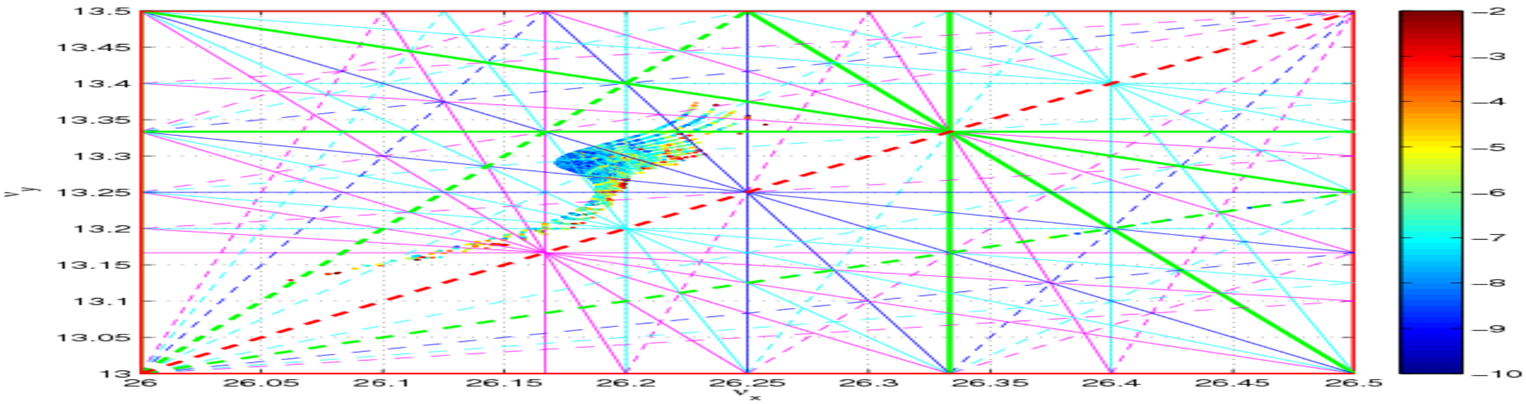


Tune diffusion rate D= $\log_{10}((\Delta\nu_x)^2+(\Delta\nu_y)^2)^{1/2}$ for tune difference between the first 512 turns and second 512 turns

Dynamic Aperture and Frequency Map Analysis

With multipole errors





Tune diffusion rate D= $\log_{10}((\Delta\nu_x)^2 + (\Delta\nu_y)^2)^{1/2}$ for tune difference between the first 512 turns and second 512 turns

Emittance

- For proton or ion beams, synchrotron radiation damping time is very long and transverse and longitudinal emittance of beam motion is constant and do not depend on lattice design. For electron and positron beams, lattice design play an important role in emittance optimization.
- For colliders, it is optimized for high luminosity at collision points.
- For synchrotron light sources, it is optimized for high brightness or brilliance at every light port, and small emittance is the most demanding parameter.
- By increasing synchrotron radiation damping rate with wigglers can help reduce natural emittance.

Equilibrium Beam Emittance

In equilibrium:

$$\frac{\sigma_x^2}{\beta_x} = \tau_x \zeta_x$$

$$\tau_x = \text{damping coefficient} = -\frac{1}{2}\frac{\langle P_\gamma \rangle}{E} J_x$$

$$\zeta_x = \text{diffusion coefficient} = \frac{55}{48\sqrt{3}} \frac{\langle P_\gamma \hbar \omega_c H_x \rangle}{E^2}$$

$$\text{with } H_x = \gamma_x D_x^2 + 2\alpha_x D_x D_x' + \beta_x D_x'^2$$

$$\text{and } P_\gamma(\text{GeV/s}) = \frac{2}{3} r_c mc^2 \frac{c\beta^4\gamma^4}{\rho^2}$$

$$\text{or more pratical units, } P_\gamma = \frac{cC_\gamma}{2\pi}\frac{E^4}{\rho^2}, C_\gamma = \frac{4\pi}{3}\frac{r_c}{(mc^2)^3} = 8.8575\times 10^{-5}\,\text{m/GeV}^3$$

$$\text{and critical energy } \varepsilon_c = \frac{3}{2}\hbar c\frac{\gamma^3}{\rho} \Rightarrow \varepsilon_c(\text{keV}) = 2.218\frac{E^3(\text{GeV})}{\rho(\text{m})} = 0.665 E^2(\text{GeV}) B(\text{T})$$

$$\text{Horizontal natural emittance } \varepsilon_x = \frac{\sigma_x^2}{\beta_x} = C_q\gamma^2\frac{\langle H_x/\rho^3\rangle}{J_x\langle 1/\rho^2\rangle}, C_q = 3.83\times 10^{-13} m$$

Equilibrium Beam Emittance

For an ideal flat accelerator without errors, due to the finite emission angle of synchrotorn radiation, we can get natural vertical emittance as:

$$\varepsilon_y = C_q \frac{\langle \beta_y / \rho^3 \rangle}{J_y \langle 1/\rho^2 \rangle} \approx 10^{-13} m - rad$$

However, due to spurious vertical dispersion by errors,

$$\varepsilon_y = C_q \gamma^2 \frac{\langle H_y / \rho^3 \rangle}{J_y \langle 1/\rho^2 \rangle} >> 10^{-13} m - rad, H_y = \gamma_y D_y^2 + 2\alpha_y D_y D_y^{'} + \beta_y D_y^{'2}$$

Minimum Emittance in FODO

For FODO Cell

$$\beta_F = \frac{2L}{\sin\Phi}(1+\sin\frac{\Phi}{2}),\ \beta_d = \frac{2L}{\sin\Phi}(1-\sin\frac{\Phi}{2}),\ \sin\frac{\Phi}{2} = \frac{L}{2f}$$

$$D_F = \frac{L\theta}{\sin^2\frac{\Phi}{2}}(1+\frac{1}{2}\sin\frac{\Phi}{2}),\ D_D = \frac{L\theta}{\sin^2\frac{\Phi}{2}}(1-\frac{1}{2}\sin\frac{\Phi}{2})$$

$$H_F = L\theta^2 \frac{\cos(\Phi/2)}{\sin^3(\Phi/2)1+\sin(\Phi/2)}(1+(1/2)\sin(\Phi/2))^2$$

$$H_D = L\theta^2 \frac{\cos(\Phi/2)}{\sin^3(\Phi/2)1-\sin(\Phi/2)}(1-(1/2)\sin(\Phi/2))^2$$

$$\langle H\rangle \approx \frac{1}{2}\rho\theta^3 \frac{\cos(\Phi/2)}{\sin^3(\Phi/2)}\left[\frac{(1+(1/2)\sin(\Phi/2))^2}{1+\sin(\Phi/2)} + \frac{(1-(1/2)\sin(\Phi/2))^2}{1-\sin(\Phi/2)}\right]$$

$$F_{FODO} = \frac{1-\frac{3}{4}\sin^2(\Phi/2)}{\sin^3(\Phi/2)\cos(\Phi/2)}/J_x$$

$$\varepsilon_x(\min) = F_{FODO} C_q \gamma^2 \theta^3,$$

where $F \sim 1.3$ for phase advance $\approx 140^\circ$ per cell.

FODO lattices are commonly used in colliders, booster synchrotrons in which emittance is not pushed to extremely small values.

Minimum Emittance in DBA

DBA:
At bend entrance,
$D_0 = D_0' = 0$
Bend length ℓ

Small angle

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$
$$D(s) = D_0 + D_0' s + \rho(1-\cos\theta) = \rho(1-\cos\theta)$$
$$D'(s) = D_0' + \sin\theta = \sin\theta$$
$$\langle H \rangle = \frac{1}{\ell}\int_0^\ell H(s)ds = \frac{1}{\ell}\int_0^\ell (\beta(s)D'^2(s) + 2\alpha(s)D(s)D'(s) + \gamma D^2(s)ds$$
$$= \rho\theta^3\left(\frac{\beta_0}{3\ell} - \frac{\alpha_0}{4} + \frac{\gamma_0\ell}{20}\right)$$
$$\varepsilon_x \cong \frac{C_q\gamma^2\theta^3}{J_x}\left(\frac{\beta_0}{3\ell} - \frac{\alpha_0}{4} + \frac{\gamma_0\ell}{20}\right)$$
$$\text{Minimum emittance}: \frac{\partial\varepsilon_x}{\partial\alpha_0} = \frac{\partial\varepsilon_x}{\partial\beta_0} = 0$$
$$\Rightarrow \beta_{0,\min} = (6/\sqrt{15})\ell, \alpha_{0,\min} = \sqrt{15}$$
$$\varepsilon_x(\min) = \frac{C_q\gamma^2\theta^3}{4\sqrt{15}J_x} \cong 0.0645\frac{C_q\gamma^2\theta^3}{J_x}$$
Minimum H occurs at
$$s^* = \frac{3}{8}\ell \text{ with } \beta^* = \frac{3}{4\sqrt{60}}\ell$$

S.Y. Lee, "Emittance optimization in three- and multiple-bend achromats", Phy. Rev. E, 52, 1940, 1996 and "Accelerator Physics", World Scientific, 2nd edition, 2004

Each MEDBA module with horizontal phase advance across dipole of 156.7^0, dispersion matching 122^0 and other. In real machine phase advance is much smaller.

Exercise 3

Minimum Emittance in DBA

For non-achromat lattice, only one type of dipole and beta and dispersion functions are symmetric at dipole center

$$\text{TME (non-achromat)} \Rightarrow \alpha(s=\ell/2)=0, D^{'}(s=\ell/2)=0$$

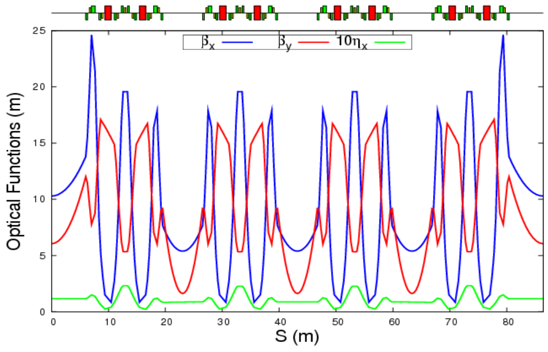
$$\varepsilon_x \cong \frac{C_q\gamma^2\theta^2}{4J_x}\left(\frac{\beta_0}{3\ell}-\frac{\alpha_0}{4}+\frac{\gamma_0\ell}{20}\right) \text{ for small } \theta$$

$$\beta_{0,\min}=\frac{8}{\sqrt{15}}\ell, \alpha_{0,\min}=\sqrt{15}, D_{0,\min}=\frac{1}{6}\rho\theta^2, D^{'}_{0,\min}=-\frac{1}{2}\theta$$

$$\varepsilon_x(TME)=\frac{C_q\gamma^2\theta^3}{12\sqrt{15}J_x}=0.0215\frac{C_q\gamma^2\theta^3}{J_x}$$

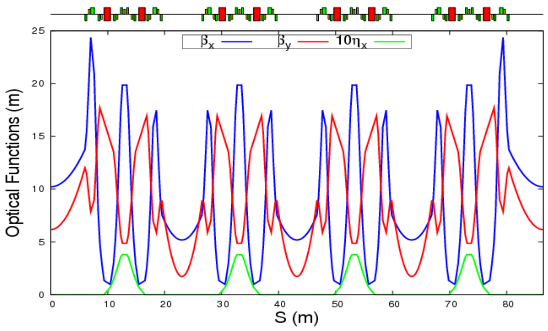
Design Emittance

- In practice, real machine will not be able to reach theoretical minimum emittance because of the constraints in betatraon function limitation, tune range, nonlinear sextupole scheme, and engineering limitation, etc.
- Usually, a few factor larger in real machine is feasible.
- For 24-cell 3GeV, theoretical minimum emittance is 1.92 nm-rad for the achromatic DBA. In real design, it can reach 4.9 nm-rad.
- For non-achromat configuration, we can reach 1.6 nm-rad (as compared with the TME of 0.64 nm-rad)



1.6 nm-rad

TPS



4.9 nm-rad

Minimum Emittance in TBA

For TBA (achromatic):

If $\frac{L_2^3}{\rho_2^2} = \frac{L_1^3}{\rho_1^2}$ and phase advance $\phi = 127.76^\circ$, i.e.,

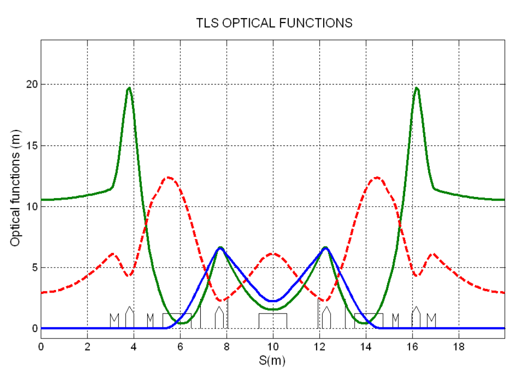
for isomagnetic ring, $L_2 = 3^{1/3} L_1$, or $\rho_1 = \sqrt{3}\rho_2$,

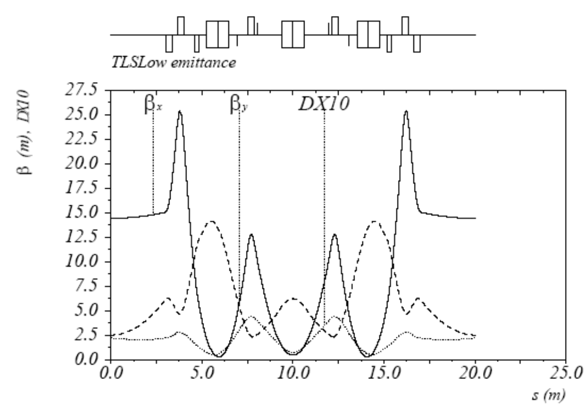
then $\varepsilon_x(\min) = \frac{1}{4\sqrt{15}} \frac{C_q \gamma^2 \theta_1^3}{J_x}$,

where L_1, θ_1 is the outer dipole length and bending radius.

But if length and field strenght are equal, then $\varepsilon_x(\min) = \frac{1.1064}{4\sqrt{15}} \frac{C_q \gamma^2 \theta^3}{J_x}$

NSRRC 1.5 GeV
6-fold TBA
Jx=1.29
TME=7.79 nm
Designed 25.6 nm





Distributed dispersion
~ 9 nm-rad

Minimum Emittance in QBA

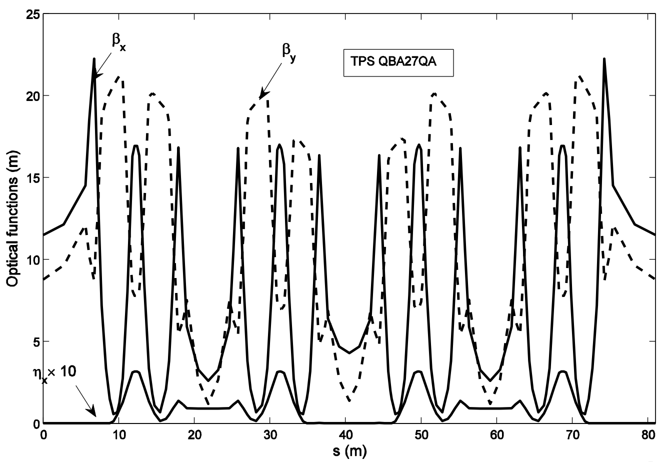
For QBA (achromatic):

If $\frac{L_2^3}{\rho_2^2} = \frac{L_1^3}{\rho_1^2}$, i.e.,

for isomagenti ce ring, $L_2 = 3^{1/3} L_1$, or $\rho_1 = \sqrt{3}\rho_2$,

then $\varepsilon_x(\min) = \frac{1}{4\sqrt{15}} \frac{C_q \gamma^2 \theta_1^3}{J_x}$,

where L_1, θ_1 is the outer dipole length and bending radius.



TPS QBA 2.98 nm-rad

If number of dipoles in DBA, TBA and QBA are the same, then

$$\epsilon_{\text{METBA}} = \left(\frac{3}{2+3^{1/3}}\right)^3 \epsilon_{\text{MEDBA}} = 0.66\epsilon_{\text{MEDBA}}$$

$$\epsilon_{\text{MEQBA}} = \left(\frac{2}{1+3^{1/3}}\right)^3 \epsilon_{\text{MEDBA}} = 0.55\epsilon_{\text{MEDBA}}$$

REVIEW OF SCIENTIFIC INSTRUMENTS **78**, 055109 (2007)

Quadruple-bend achromatic low emittance lattice studies

M. H. Wang, H. P. Chang, H. C. Chao, P. J. Chou, C. C. Kuo, and H. J. Tsai
National Synchrotron Radiation Research Center, Hsinchu 30076, Taiwan
S. Y. Lee,[a)] W. M. Tam, and F. Wang
Department of Physics, Indiana University, Bloomington, Indiana 47405

When the storage ring operates at the nonachromatic mode, the dispersion function is not zero in the ID straight sections, and thus the beam momentum spread can contribute to the horizontal beam width. We define the "one dimensional (1D) effective emittance" in the ID section as

$$\epsilon_{x,\mathrm{1D}} = \epsilon_x + \mathcal{H}_{\mathrm{ID}}\sigma_\delta^2, \tag{3}$$

where $\mathcal{H}_{\mathrm{ID}}$ is the $\mathcal{H}$-function at the ID locations, $\sigma_\delta^2 = C_q\gamma^2/(\mathcal{J}_E\rho)$ is the square of the rms momentum spread, and $\mathcal{J}_E=2+I_4/I_2\approx 2$ is the longitudinal damping-partition number. More commonly, in the light source community, the effective emittance is defined as

$$\epsilon_{x,\mathrm{eff}} = \sqrt{\epsilon_x\epsilon_{x,\mathrm{1D}}}. \tag{4}$$

Since it is difficult to design a lattice reaching either the MEDBA or the TME condition, we ask a simple question: are we better off with the achromatic or the nonachromatic mode of operation? Let us assume we can design a lattice in the achromatic mode with $\epsilon_{x,\mathrm{a}}=f_\mathrm{a}\epsilon_{\mathrm{MEDBA}}$ and another nonachromatic mode with $\epsilon_{x,\mathrm{na}}=f_{\mathrm{na}}\epsilon_{\mathrm{TME}}$, where both f_a and f_{na} are typically about 2–4. The effective emittances of these two lattices become

$$\epsilon_{x,\mathrm{a}} = \frac{f_\mathrm{a}}{4\sqrt{15}\mathcal{J}_x}C_q\gamma^2\theta^3, \tag{5}$$

$$\epsilon_{x,\mathrm{1D,na}} = \frac{f_{\mathrm{na}}}{12\sqrt{15}\mathcal{J}_x}C_q\gamma^2\theta^3 + \frac{\rho\theta^3}{3\sqrt{15}}\frac{C_q\gamma^2}{\mathcal{J}_E\rho}, \tag{6}$$

where we have used $\mathcal{H}_{\mathrm{ID}}\approx\mathcal{H}_{\mathrm{TME}}$. Assuming $\mathcal{J}_x\approx 1$ and $\mathcal{J}_E\approx 2$, we find

$$\frac{\epsilon_{x,\mathrm{1D,na}}}{\epsilon_{x,\mathrm{a}}} \approx \frac{f_{\mathrm{na}}+2}{3f_\mathrm{a}}, \quad \frac{\epsilon_{x,\mathrm{eff,na}}}{\epsilon_{x,\mathrm{a}}} \approx \frac{\sqrt{f_{\mathrm{na}}(f_{\mathrm{na}}+2)}}{3f_\mathrm{a}}. \tag{7}$$

If both f_a and f_{na} are 2, the 1D and effective emittance of the nonachromatic lattice are about about 0.66 and 0.47 of the achromatic one, respectively, i.e., the nonachromatic lattice can provide *higher beam brilliance*. However, the emittance reduction factor is not 1/3.

Effective emittance

With dispersion in the long straights, we can reduce the natural emittance by a factor of 3, but the effective emittance is

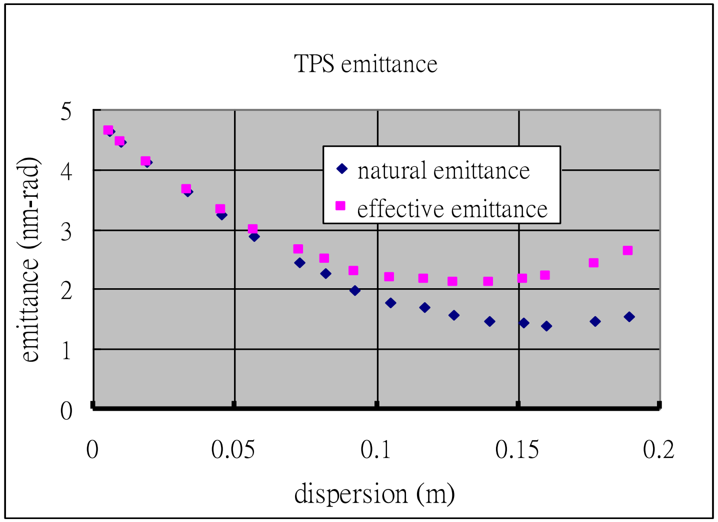
$$\varepsilon_{x,eff}{}^{2} = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2$$

$$\varepsilon_{x,eff} = \sqrt{\varepsilon_x^2 + H_{ID}\sigma_E^2\varepsilon_x}$$

$$H_{ID} = \gamma_x D_x^2 + 2\alpha_x D_x D_x' + \beta_x D_x'^2$$

At the symmetry point of the long straights,

$$\varepsilon_{eff} = \varepsilon_x\sqrt{1+\frac{(\sigma_E D_x)^2}{\varepsilon_x\beta_x}}$$



Closed Orbit Distortion

In reality, dipole field errors distributed around the ring:

$$y_{co}(s) = \int_s^{s+c} G(s,s') \frac{\Delta B(s')}{B\rho} ds'$$

$$\text{where} \quad G(s,s') = \frac{\sqrt{\beta(s)\beta(s')}}{2\sin\pi\nu} \cos(\pi\nu - |\psi(s) - \psi(s')|)$$

In dipole angular kick form:

$$y_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi\nu} \sum_{i=1}^{N} \theta_i \sqrt{\beta(s_i)} \cos(\pi\nu - |\psi(s) - \psi(s_i)|) \qquad \theta_i = \frac{\Delta B(s_i)}{B\rho} ds_i$$

$$y_{co,rms} = \frac{\sqrt{\beta(s)\overline{\beta}}}{2\sqrt{2}\,|\sin\pi\nu|} \sqrt{N}\theta_{rms}$$

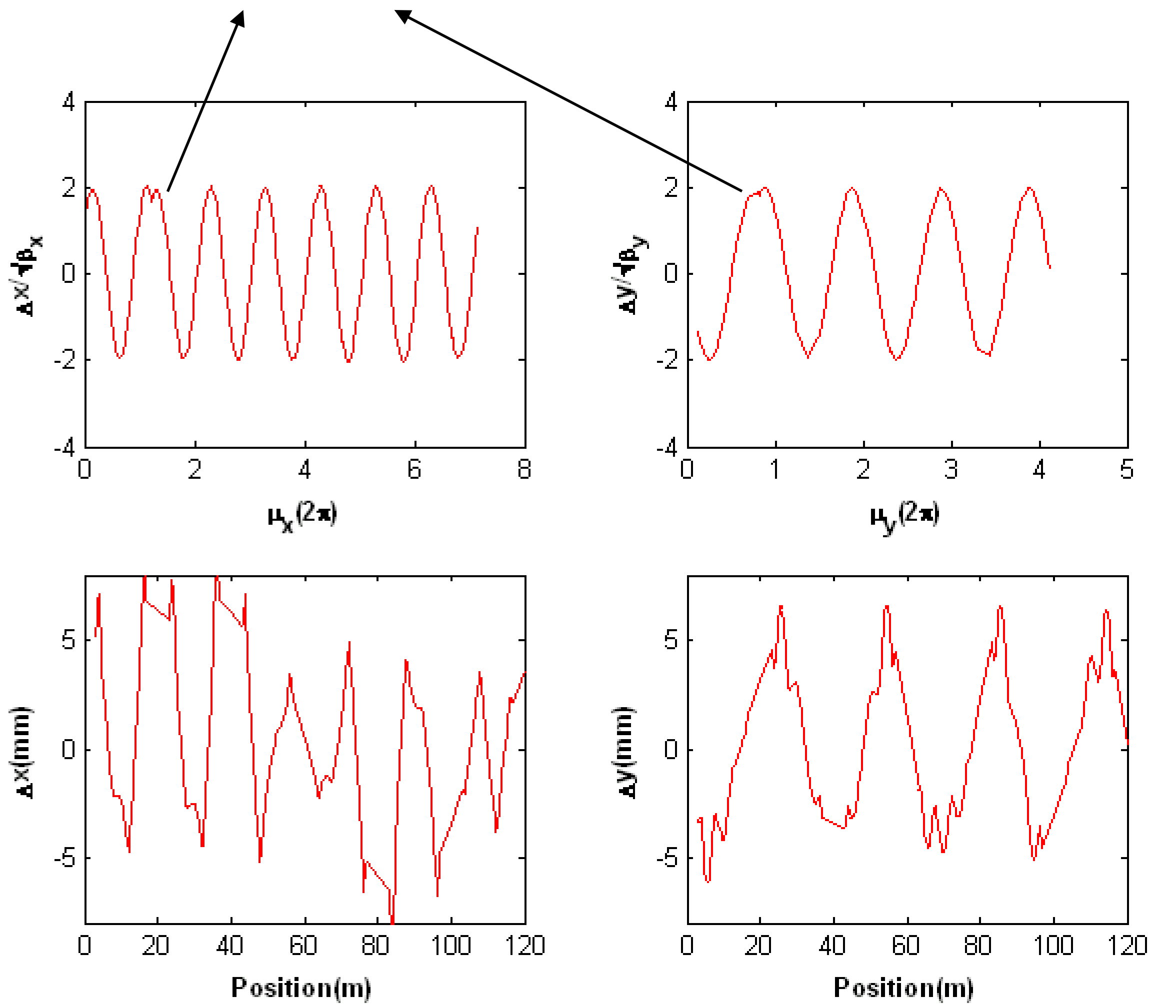
due to quadrupole misalignment

$$\theta = \frac{(\partial B_z / \partial x)\ell}{B\rho} \Delta y = \frac{\Delta y}{f}$$

$$y_{co,rms} \approx \left\{ \frac{\beta_{\text{av}}}{2\sqrt{2} f_{\text{av}} |\sin\pi\nu|} \sqrt{N_q} \right\} \Delta y_{rms,}$$

The coefficient in curly brackets is called sensitivity or amplification factor.

“cusp” where error kinks take place(0.5mrad in x and 1 mrad in y)

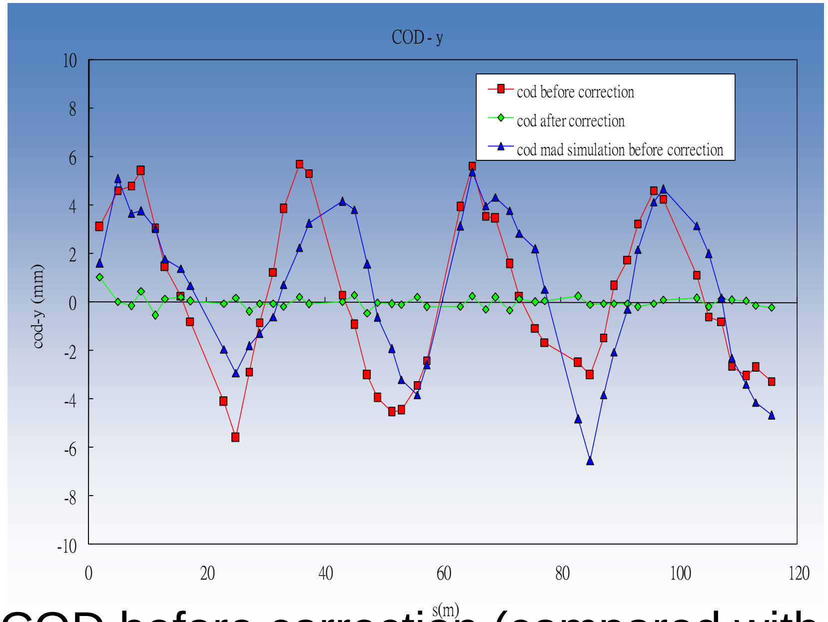


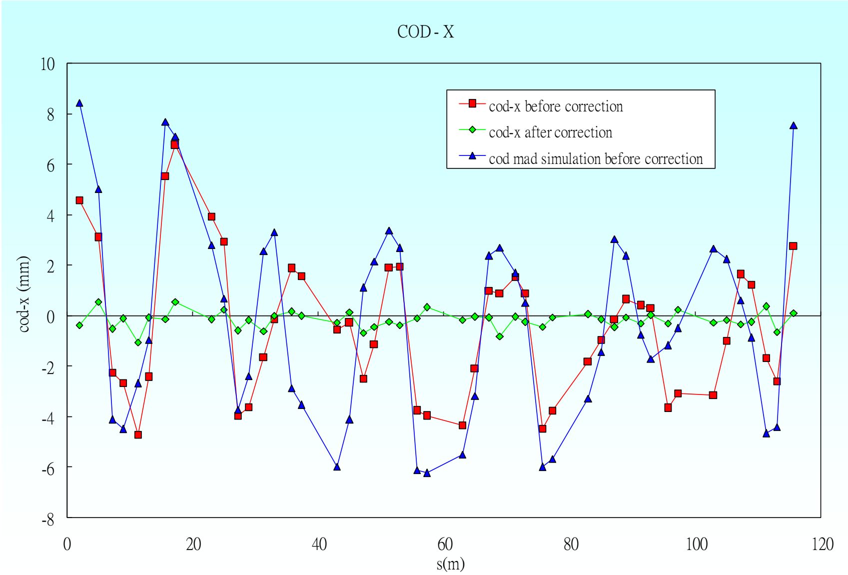
Orbit distortions due to single dipole error kick in x and y planes in TLS ring, working point (7.18, 4.13) without errors.

An example in TLS storage ring:

$$x^2_{co,rms}(mm) = (32.3)^2 \Delta x^2_{q,rms}(mm) + (3.5)^2 (\frac{\Delta B\ell}{B\ell})_{rms}{}^2 (10^{-3})$$

$$z^2_{co,rms}(mm) = (22.6)^2 \Delta z^2_{q,rms}(mm) + (9.3)^2 (\Delta\theta_{s,BM})_{rms}{}^2 (mrad)$$





COD before correction (compared with model simulation with errors input) in SRRC Storage ring at commissioning stage in1993. Corrected COD is shown too.

Closed Orbit Correction Methods:

Harmonic method, Beam-bump method. Least-squares method, Singular value decomposition method, etc.

SVD (Singular Value Decomposition) method:

for any rectangular matrix A[m x n] with rank k,

there exist unitary matrices U[m x m] and V[n xn] such that

$$A = UWV^T$$

$$UU^T = U^T U = VV^T = V^T V = 1 \text{ orthogonal}$$

$$U^{-1} = U^T \,,\; V^{-1} = V^T$$

$$\vec{y} = -A\vec{\theta} \Rightarrow \vec{\theta} = -A^{-1}\vec{y} = -VW^{-1}U^T\vec{y}$$

$$\vec{\theta} = -V\left(diag(1/w_j)\right)U^T \cdot \vec{y}$$

$$\text{If } 1/w_j > \text{threshold, let } 1/w_j = 0,$$

$$\text{non-zero up to k}$$

rms strength of correctors reduced—corrector ironing, good for the reduction of vertical dispersion

COD correction, SVD method:

$$\Delta y_{co}(s_j) = \frac{\sqrt{\beta(s_j)}}{2\sin\pi\nu}\sum_{i=1}^{N}\theta_i\sqrt{\beta(s_i)}\cos(\pi\nu - |\psi(s_j)-\psi(s_i)|)$$

$$A_{ji} = \frac{\sqrt{\beta(s_j)}}{2\sin\pi\nu}\sqrt{\beta(s_i)}\cos(\pi\nu - |\psi(s_j)-\psi(s_i)|)$$

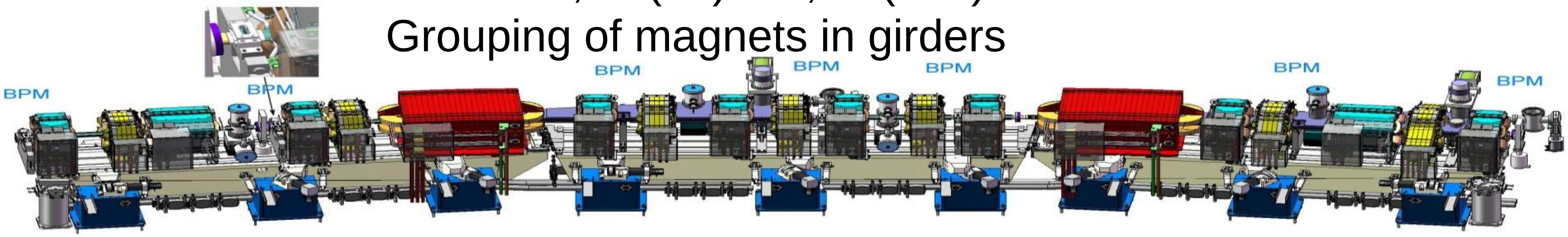
The distorted orbit can be minimized at least at the BPM to the desired value so that

$$\vec{\theta}_n = -A^{-1}\Delta\vec{y}_{co,m}$$

SVD algorithm is employed in the simulations.

COD Correction Scheme at TPS

168 BPM, 72(96) HC, 96(120) VC
Grouping of magnets in girders



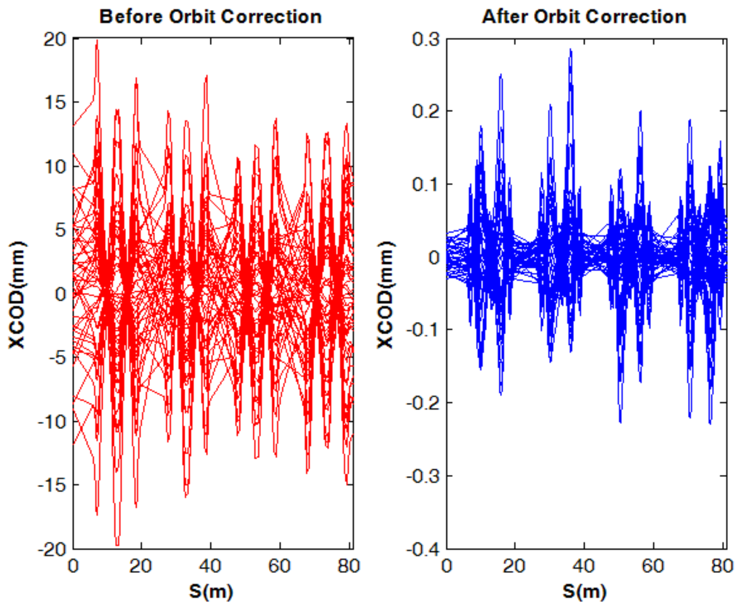
Error Source (rms) 3 sigma truncated	
Girder displacement x, y (mm)	0.1
Girder roll θ (mrad)	0.1
Quad and sext displacement x,y w.r.t. girder (mm)	0.03
Dipole displacement x,y (mm)	0.5
Dipole roll θ (mrad)	0.1
Dipole field error (10^{-3})	1
BPMs displacement x, y (mm)	0.1

Amplification factor	Ax rms (max)	Ay rms (max)
Quad displacement	55 (97)	40 (51)
Girder displacement	30 (54)	8 (10)
Dipole roll θ	-	5.8 (7.8)
Dipole field error	1.1 (1.9)	-

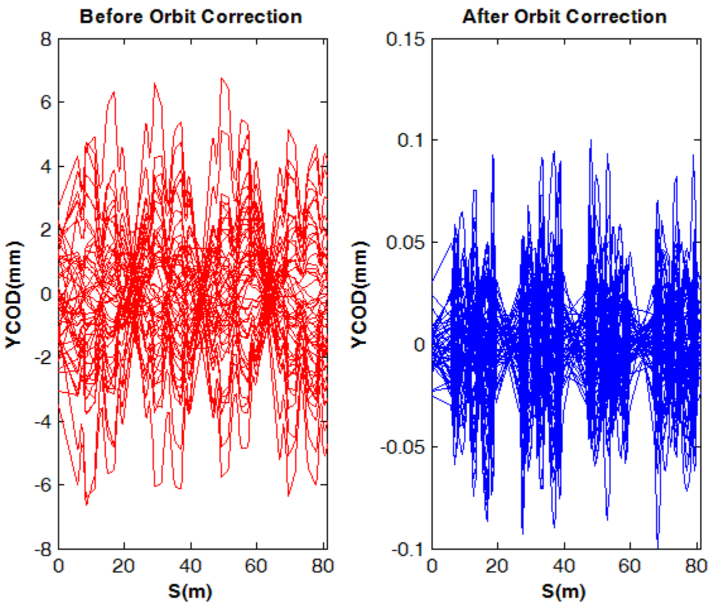
7 BPM each cell
3 HC(+1) and 4 VC(+1) each cell for SVD
but all (7) sextupoles are with HC and VC.

COD due to Errors:
Horizontal: 3.8 mm rms
Vertical : 2.2 mm rms
After correction:
Horizontal: 0.08 mm rms
Vertical : 0.06 mm rms

TPS COD



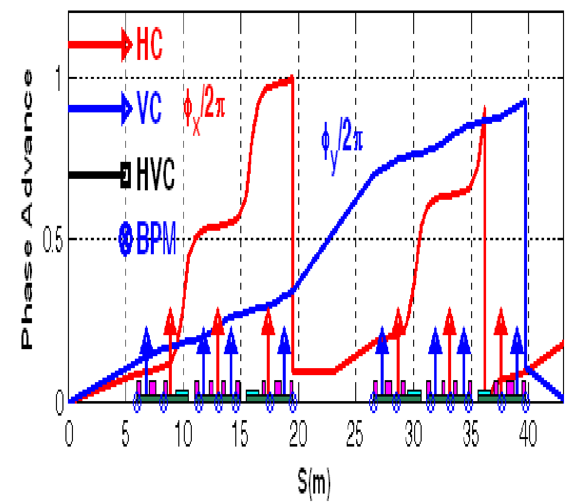
a) Horizontal COD before and after correction in one superperiod in TPS for 50 random machines.

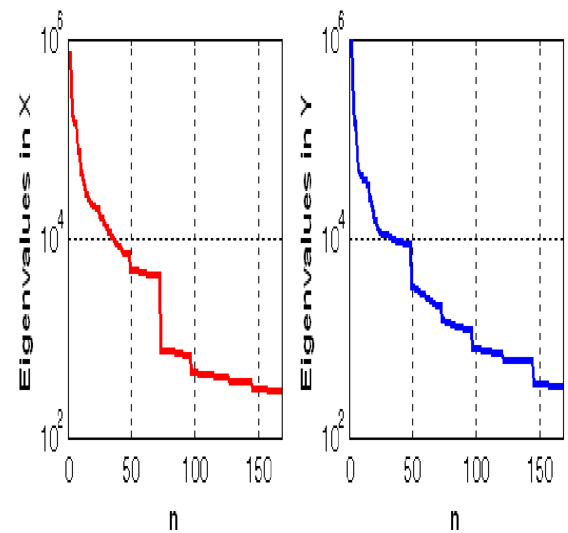


b) Vertical COD before and after correction in one superperiod in TPS for 50 random machines

Correction Capability and Residual COD

	Correctors Used	Number of eigenvalues used	Mean of <|cor. Str.|> (mrad)	Max of |cor. Str.| (mrad)	Max |COD| at BPM (mm)	rms COD at BPM (mm)
Horizontal	(2,4,6)	72	7.88E-02	4.02E-01	3.71E-01	8.07E-02
	(1,4,6)	72	7.34E-02	3.97E-01	3.46E-01	8.35E-02
	(2,4,7)	72	7.35E-02	4.34E-01	3.66E-01	8.32E-02
	(1,4,7)	72	6.69E-02	3.41E-01	3.67E-01	8.55E-02
	168, (C1-C7)x24	72	3.20E-02	1.70E-01	3.53E-01	8.17E-02
		96	5.44E-02	4.35E-01	2.92E-01	6.87E-02
		144	1.22E-01	7.93E-01	2.12E-01	4.06E-02
		168	1.63E-01	9.73E-01	4.92E-02	7.71E-03
Vertical	(1,3,5,7)	96	4.84E-02	2.45E-01	3.38E-01	6.76E-02
	(2,3,5,7)	96	5.64E-02	3.51E-01	3.36E-01	7.18E-02
	168, (C1-C7)x24	48	1.35E-02	8.73E-02	3.95E-01	9.23E-02
		72	1.98E-02	1.43E-01	3.42E-01	7.97E-02
		96	3.08E-02	1.92E-01	3.10E-01	6.82E-02
		144	7.21E-02	4.43E-01	2.99E-01	4.13E-02
		168	1.10E-01	8.95E-01	7.52E-02	1.43E-02





ID effect

Insertion devices cause some effects on beam dynamics like betatron tune shifts, optical functions perturbation, emittance variation, multipole field effects, etc. The field of a wiggler can be modeled as:

$$B_x = 0 \qquad B_y = B_w \cos(k_w s)\cosh(k_w y) \qquad B_s = -B_w \sin(k_w s)\sinh(k_w y)$$

The equations of motion:

$$\begin{cases} \dfrac{d^2x}{ds^2} = \dfrac{1}{\rho_w}\cosh(k_w y)\cos(k_w s), \\ \dfrac{d^2y}{ds^2} + \dfrac{\sin^2(k_w s)}{\rho_w^2}\dfrac{\sinh(2k_w y)}{2k_w} = \dfrac{p_x}{\rho_w}\sinh(k_w y)\sin(k_w s), \end{cases}$$

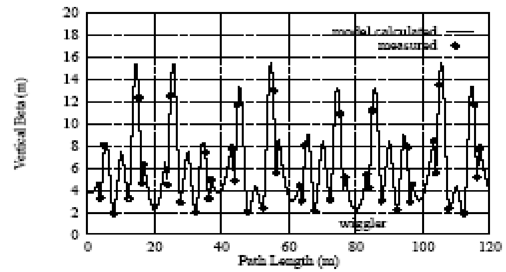


Figure 2: Measured and model calculated vertical betatron function with wiggler gap closed to 23.2 mm. The wiggler center is at 80 m from the injection point.

average focusing strength: $\left\langle \dfrac{\sin^2(k_w s)}{\rho_w^2} \right\rangle = \dfrac{1}{2\rho_w^2}$

vertical tune shift: $\Delta\nu_y = \dfrac{\bar{\beta}_y L_w}{8\pi\rho_w^2}$

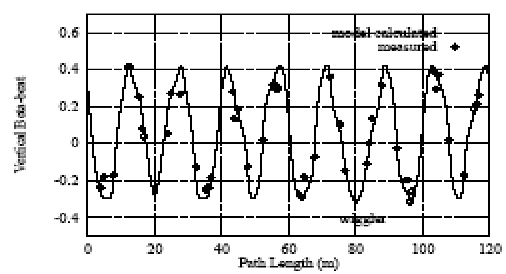
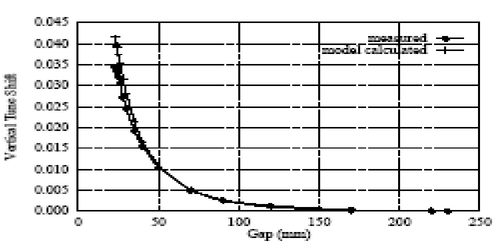


Figure 3: The measured and model calculated beta-beat between no wiggler and with wiggler gap 23.2 mm.

Beta beat in vertical plan can be evaluated



Tune-shift with gap for w20 at TLS

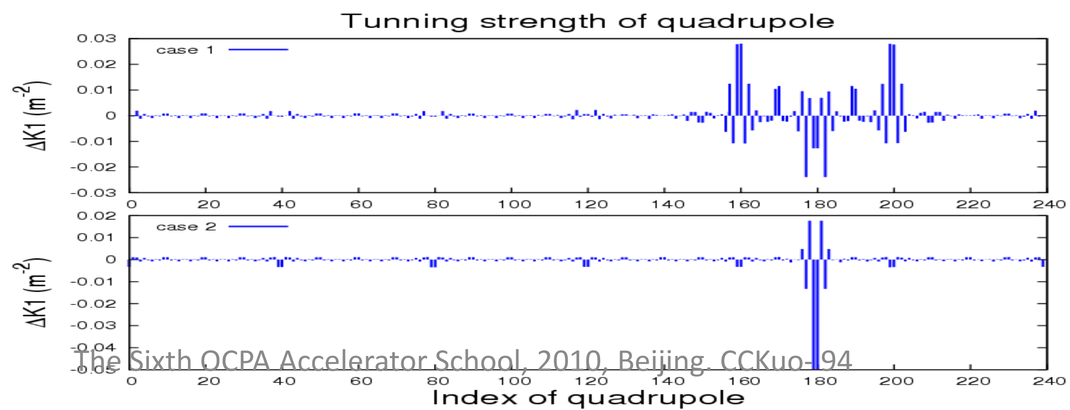
Vertical beta perturabtion due to Wiggler 20 in TLS (Kuo, et al PAC95)

Correction Algorithm

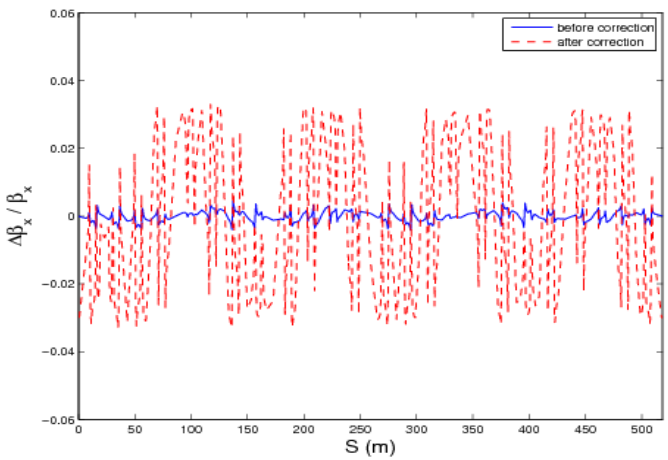
1. By the following linear relations, construct the response matrix A of bare lattice.
2. Use SVD method to restore the perturbed optics back to the optics of the bare lattice as much as possible by minimizing (AΔk +b).
3. Where Δk is the tuning strength of the quadrupoles, and b is the perturbed optics.
4. One can choose arbitrary positions to restore optics with different weighting.

$$\begin{pmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_n \end{pmatrix} \begin{pmatrix} \frac{\Delta\beta_{x1}}{\beta_{x1}} \\ \vdots \\ \frac{\Delta\beta_{y1}}{\beta_{y1}} \\ \vdots \\ \Delta\mu_{x1} \\ \vdots \\ \Delta\mu_{y1} \\ \vdots \\ \Delta Q_x \\ \Delta Q_y \end{pmatrix} = A \begin{pmatrix} \Delta k_1 \\ \Delta k_2 \\ \vdots \\ \Delta k_{nq} \end{pmatrix}$$

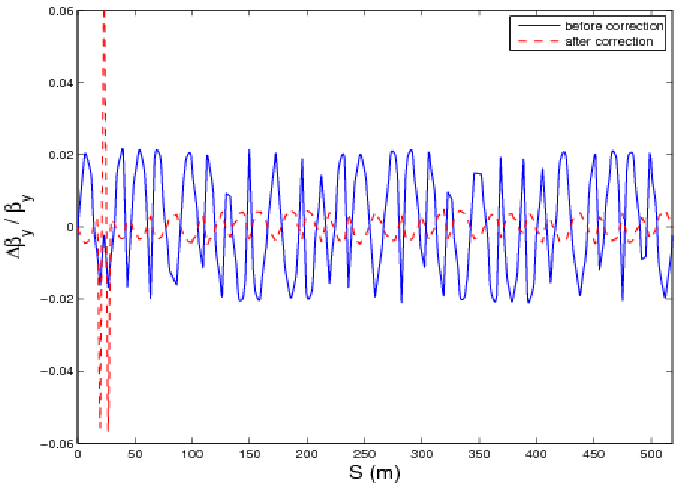
	SW60
Current (A)	0.4
λ (mm)	60
N_{period}	8
By (T)	3.5
Bx (T)	
L (m)	0.45
Gap (mm)	17
Total power (kW)	13.39



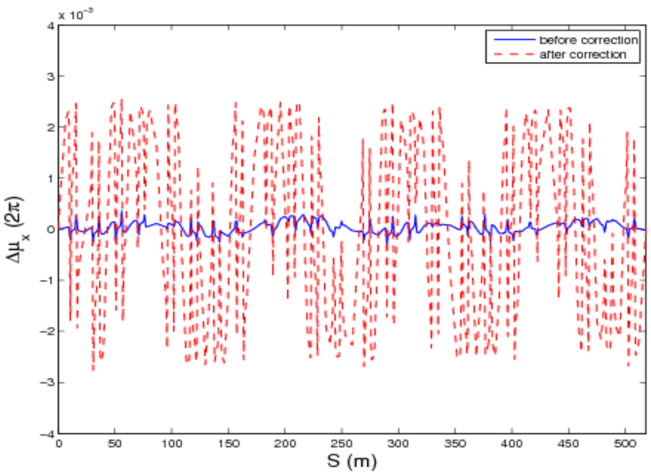
Beta Beating and phase beating correction



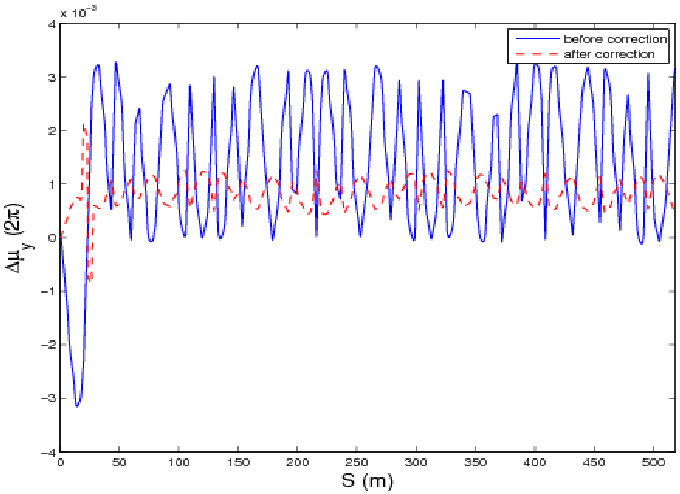
(a) Horizontal beta-beating with SW60



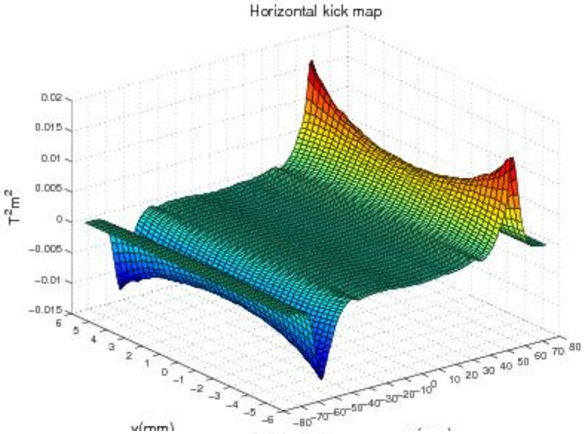
(b) Vertical beta-beating with SW60

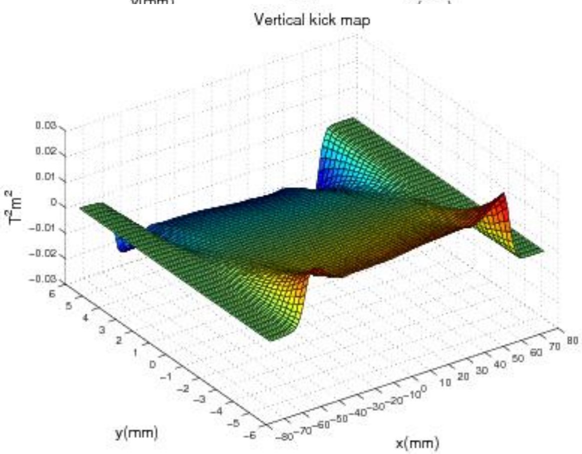


(c) Horizontal phase-beating with SW60



(d) Vertical phase-beating with SW60



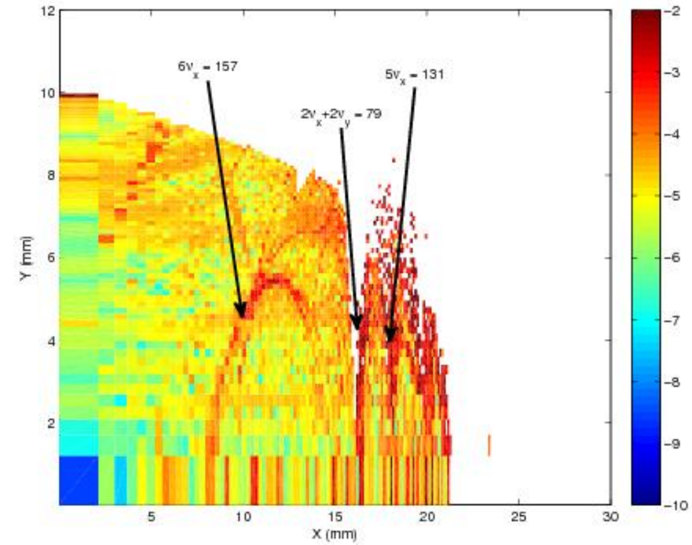


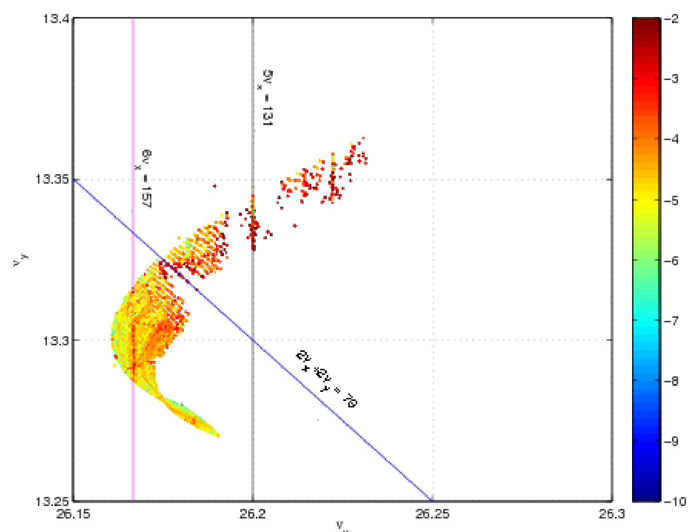
potential function:

$$\Psi(x,y)=\int_{-\infty}^{\infty}\left(\left(\int_{-\infty}^{s}B_x(x,y,s_1)ds_1\right)^2+\left(\int_{-\infty}^{s}B_y(x,y,s_1)ds_1\right)^2\right)ds$$

kick map:

$$\begin{cases} M_x=-\frac{1}{2}\partial_x\Psi(x,y) \\ M_y=-\frac{1}{2}\partial_y\Psi(x,y) \end{cases}$$





Dynamic aperture and Frequency map with SW60

Emittance with IDs

Emittance without ID $\varepsilon_{x0} = C_q \gamma^2 \frac{I_{50}}{I_{20} - I_{40}}$

Emittance with ID $\varepsilon_x = C_q \gamma^2 \frac{I_{50} + I_{5w}}{I_{20} - I_{40} + I_{2w} - I_{4w}}$

$$I_{50} \approx f_h H_{ID} \oint \frac{1}{\rho^3} ds = f_h H_{ID} \frac{2\pi}{\rho_0^2},$$

$$f_h \equiv \frac{<H_{dipole}>}{H_{ID}}, \quad H = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_x' + \beta_x \eta_x'^2$$

$$I_{5w} = H_{ID} \int_w \frac{1}{|\rho|^3} ds \approx \begin{cases} 4H_{ID} L_w / (3\pi \rho_w^3), & \text{planar undulator} \\ H_{ID} L_w / \rho_w^3, & \text{helical undulator} \end{cases}$$

$$U_0 = C_\gamma E^4 / \rho_0$$

$$U_w = \begin{cases} C_\gamma E^4 L_w / (4\pi \rho_w^2), & \text{planar undulator} \\ C_\gamma E^4 L_w / (2\pi \rho_w^2), & \text{helical undulator} \end{cases}$$

$$\frac{I_{5w}}{I_{50}} \approx \begin{cases} \frac{8\rho_0}{3\pi f_h \rho_w} \frac{U_w}{U_0} = \frac{8B_w}{3\pi f_h B_0} \frac{U_w}{U_0}, & \text{planar undulator} \\ \frac{\rho_0}{f_h \rho_w} \frac{U_w}{U_0} = \frac{B_w}{f_h B_0} \frac{U_w}{U_0}, & \text{helical undulator} \end{cases}$$

$$\frac{\varepsilon_x}{\varepsilon_{x0}} \approx \begin{cases} \dfrac{(1 + \sum_w \frac{8B_w}{3\pi f_h B_0} \frac{U_w}{U_0})}{(1 + \sum_w \frac{U_w}{U_0})}, & \text{planar undulators} \\ \dfrac{(1 + \sum_w \frac{B_w}{f_h B_0} \frac{U_w}{U_0})}{(1 + \sum_w \frac{U_w}{U_0})}, & \text{helical undulators} \end{cases}$$

➔ **If $B_w > (3\pi f_h/8)B_0$ then emittance will increase for planar undulator**

➔ **Installing IDs in the non-dispersive straights, where f_h is very large, emittance always decreases.**

➔ **For a TME lattice, f_h=0.25, and for a tyical well-designed distributed dispersion lattice, f_h=0.5~0.8**

Energy spread with IDs

$$I_{30} = \frac{2\pi}{\rho_0^2}, \quad I_{3w} = \begin{cases} 4L_w/(3\pi\rho_w^3), & \text{planar undulator} \\ L_w/\rho_w^3, & \text{helical undulator} \end{cases}$$

$$\frac{I_{3w}}{I_{30}} \approx \begin{cases} \dfrac{8\rho_0}{3\pi\rho_w}\dfrac{U_w}{U_0} = \dfrac{8B_w}{3\pi B_0}\dfrac{U_w}{U_0}, & \text{planar undulator} \\ \dfrac{\rho_0}{f\rho_w}\dfrac{U_w}{U_0} = \dfrac{B_w}{B_0}\dfrac{U_w}{U_0}, & \text{helical undulator} \end{cases}$$

$$U_0 = C_\gamma E^4/\rho_0$$

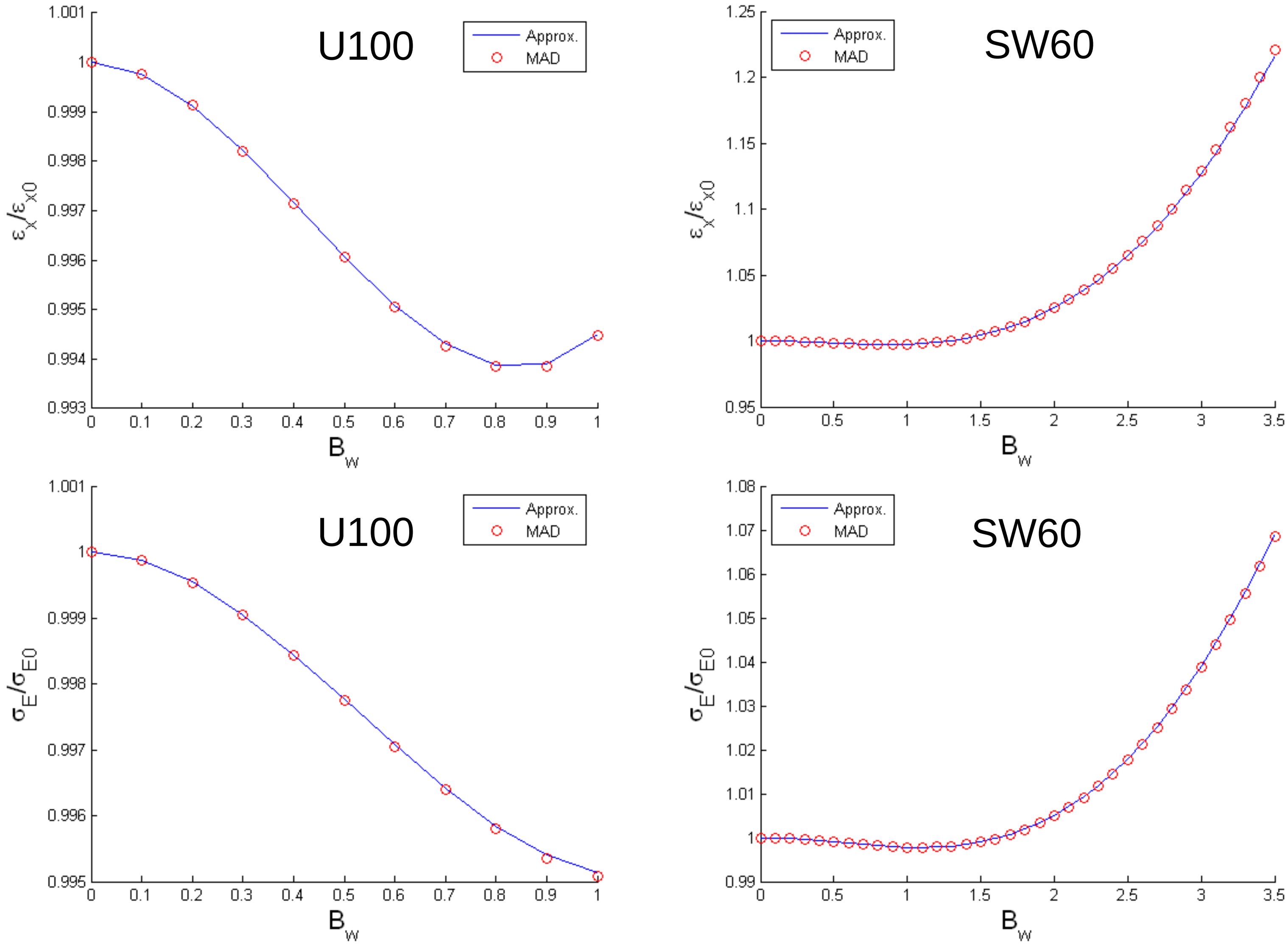
$$U_w = \begin{cases} C_\gamma E^4 L_w/(4\pi\rho_w^2), & \text{planar undulator} \\ C_\gamma E^4 L_w/(2\pi\rho_w^2), & \text{helical undulator} \end{cases}$$

$$\left(\frac{\sigma_E}{E}\right)^2 = C_q\gamma^2\frac{I_3}{(I_2+I_4)} = \left(\frac{\sigma_E}{E}\right)_0^2\frac{1+I_{3w}/I_{30}}{1+I_{2w}/I_{20}}$$

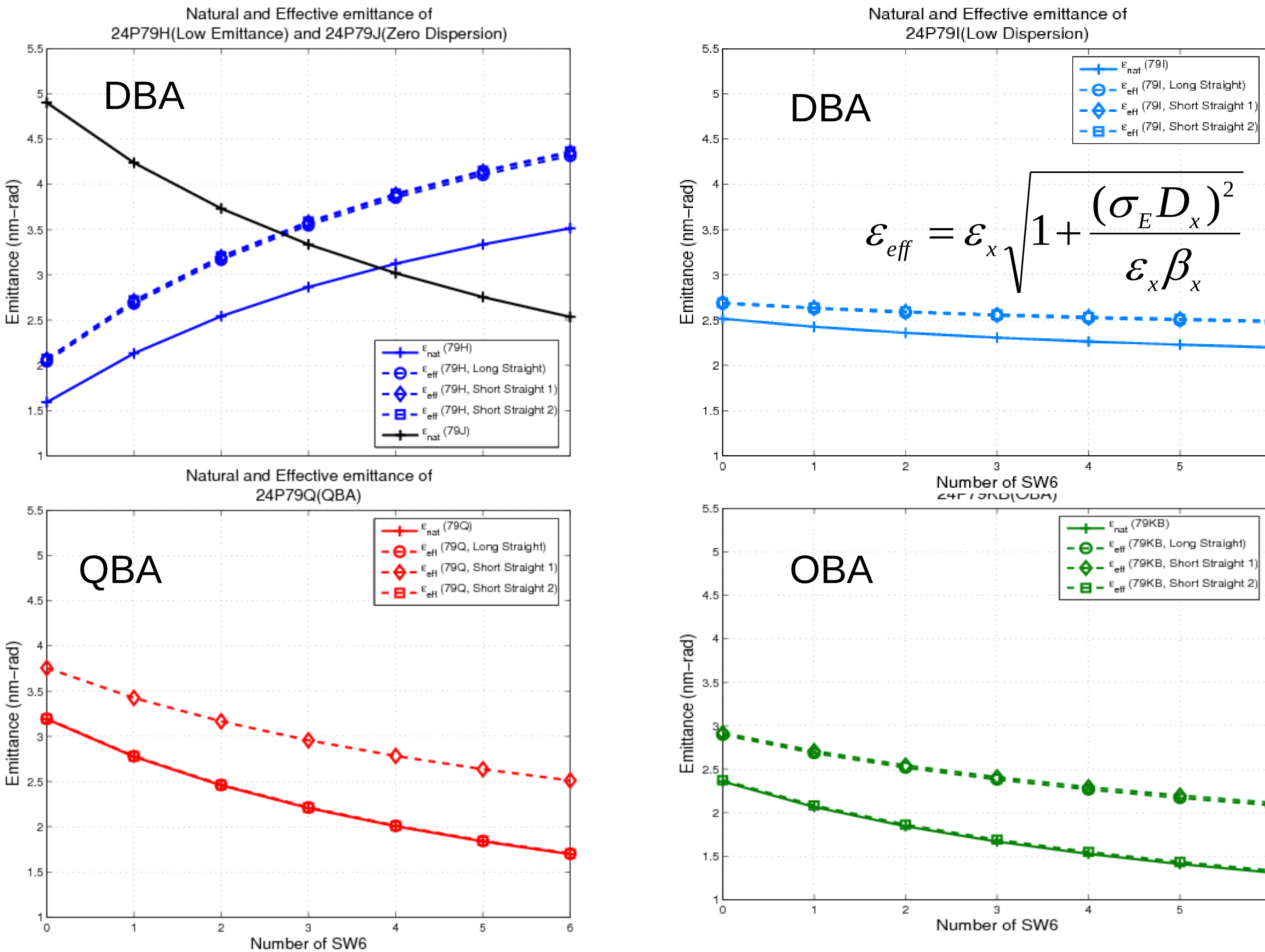
$$\left(\frac{\sigma_E}{E}\right)^2 = \left(\frac{\sigma_E}{E}\right)_0^2 \times \begin{cases} \left(1+\sum_w \dfrac{8B_w}{3\pi B_0}\dfrac{U_w}{U_0}\right)/\left(1+\sum_w\dfrac{U_w}{U_0}\right), & \text{planar undulators} \\ \left(1+\sum_w \dfrac{B_w}{B_0}\dfrac{U_w}{U_0}\right)/\left(1+\sum_w\dfrac{U_w}{U_0}\right), & \text{helical undulators} \end{cases}$$

➔If $B_w > (3\pi/8)B_0$ then energy spread will increase for planar undulator.

ID effects


U100
Approx.
MAD
SW60
Approx.
MAD
U100
Approx.
MAD
SW60
Approx.
MAD
B_W
εx/εx0
σE/σE0

Natural emittance and effective emittance as a function of number of SW60 (2 m, 3.5 T, 6 cm period length, 62.4 kW SR loss/each @400 mA)



$$\varepsilon_{eff} = \varepsilon_x \sqrt{1 + \frac{(\sigma_E D_x)^2}{\varepsilon_x \beta_x}}$$

Error sources, which cause the vertical emittanc blow–up:

- **Collective instabilities such as fast-ion instabilities, single-bunch broad-band instabilities, coupled-bunch instabilities, etc.**

- **Linear betatron coupling due to skew quadrupole errors from quadrupole rotation errors and vertical closed orbit distortion in sextupoles.**

- **Linear betatron coupling from solenoidal field.**

- **Spurious vertical dispersion caused by**
  - ✓**vertical bend error from bending rotation errors and vertical closed orbit errors in the quadrupoles**
  - ✓**dispersion coupling due to skew quadrupole errors in the dispersion region which are from quadrupole rotation errors in the dispersion region and vertical closed orbit distortion in sextupoles in the dispersion region.**

Spurious Vertical Dispersion

$$\varepsilon_y = \frac{c_q \gamma^2 \langle I_y \rangle}{\rho} = \frac{c_q \gamma^2}{\rho} \left\langle [\eta_y^2 + (\eta_y \alpha_y + \eta_y ' \beta_y)^2] / \beta_y \right\rangle_{dipole} \approx 2 \frac{c_q \gamma^2}{\rho} \left\langle \eta_y^2 / \beta_y \right\rangle_{dipole}$$

$$\begin{aligned} \left\langle \eta_y^2 / \beta_y \right\rangle_{dipole} &= \frac{1}{8 \sin^2 \pi \nu_y} \left(\sum_i \frac{\beta_{yi} L_i^2}{\rho^2} \Delta \vartheta_{i,dipole}^2 + \sum_i \beta_{yi} k_{1i}^2 L_i^2 \Delta y_{cod,i,quad}^2 \right. \\ &\quad \left. + \sum_i \beta_{yi} k_{1i}^2 \eta_{xi}^2 L_i^2 \Delta \vartheta_{i,quad}^2 + \sum_i \beta_{yi} k_{2i}^2 \eta_{xi}^2 L_i^2 \Delta y_{cod,i,sext}^2 \right) \\ &= \frac{1}{8 \sin^2 \pi \nu_y} \sum_i \beta_{yi} F_i^2 L_i^2, \end{aligned}$$

$$F_i = \begin{Bmatrix} \dfrac{\Delta \vartheta_{i,dipole}}{\rho} \\ k_{1i} \Delta y_{cod,i,quad} \\ k_{1i} \eta_{xi} \Delta \vartheta_{i,quad} \\ k_{2i} \eta_{xi} \Delta y_{cod,i,sext} \end{Bmatrix}$$

TPS:

$$\varepsilon_{\mathrm{y}}(nm-rad) = 4.3\times 10^{-3}\, \Delta \vartheta_{dipole}^2 + 3.98\times 10^{-1} \Delta y_{cod,\,quad}^2 + 6.3\times 10^{-3} \Delta \vartheta_{quad}^2 + 1.4\times 10^{-1} \Delta y_{cod,\,sext}^2$$

Betatron coupling

$$G_{1,-1,l}e^{i\chi} = \frac{1}{2\pi}\oint \sqrt{\beta_x\beta_y}k_s e^{i[\varphi_x-\varphi_y-(\nu_x-\nu_y-l)\theta]}ds$$

$$\left\langle G^2 \right\rangle = \frac{1}{(2\pi)^2}\left[\Delta\vartheta^2_{quad}\sum_i (2k_1 l)^2 \beta_{x,i}\beta_{y,i} + \Delta y^2_{cod,set,}\sum_i (k_2 l)^2 \beta_{x,i}\beta_{y,i}\right]$$

$$G^2 = 7.71\times10^{-4}\Delta\vartheta^2_{quad} + 3.55\times10^{-3}\,\Delta y^2_{cod,sex}$$

$$\kappa = \frac{G^2}{G^2 + 2\Delta^2}$$

Betatron coupling from quadrupoles and sextupoles

	Δ	G	K
Quadrupole Rotation: 0.1 mrad (rms)	9.94E-02	1.39E-03	3.9E-02
Vertical Sextupole Position: (rms)	9.94E-02	2.98E-03	1.79E-01

Cross Orbit Response Matrix

Vertical orbit and dispersion response to errors

$$y_c(s) = \frac{\sqrt{\beta_y(s)}}{2\sin\pi\nu_y}\int_s^{s+c}\sqrt{\beta_y(z)}\cos(\psi_y(s)-\psi_y(z)+\pi\nu_y)G(z)dz,$$

$$\eta_y(s) = \frac{\sqrt{\beta_y(s)}}{2\sin\pi\nu_y}\int_s^{s+c}\sqrt{\beta_y(z)}\cos(\psi_y(s)-\psi_y(z)+\pi\nu_y)F(z)dz,$$

$$G(s) = \tilde{K}_1 x_c + K_2 x_c y_m - G_y$$

$$F(s) = -G_y - K_1 y_c - \tilde{K}_1 \eta_x + K_2 y_c \eta_x$$

Response Matrix

skew quads and sextupoles only

$$(\Delta y_c)_i = \sum_s (K_2 l y_m)_s R_{is}^{(V)} \cdot R_{sj}^{(H)} \cdot \theta_{Hj} + \sum_k (\tilde{K}_1 l)_k R_{ik}^{(V)} \cdot R_{kj}^{(H)} \cdot \theta_{Hj},$$

$$(\Delta \eta_y)_i = \sum_s (K_2 l y_m)_s R_{is}^{(V)} \eta_x - \sum_k (\tilde{K}_1 l)_k R_{ik}^{(V)} \eta_x,$$

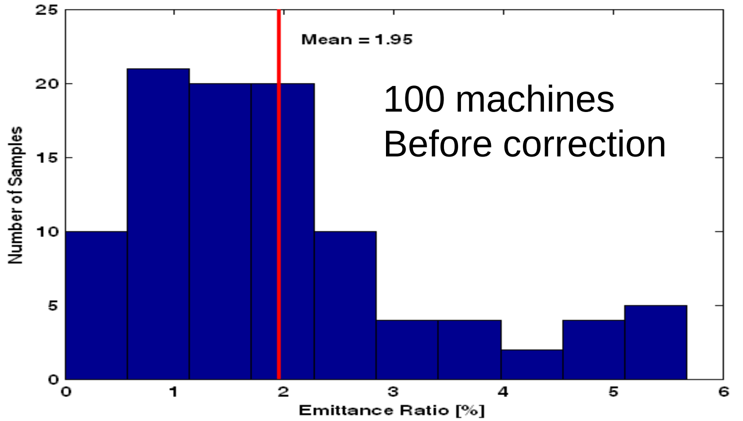
$$\mathbf{MK} = \mathbf{V}$$

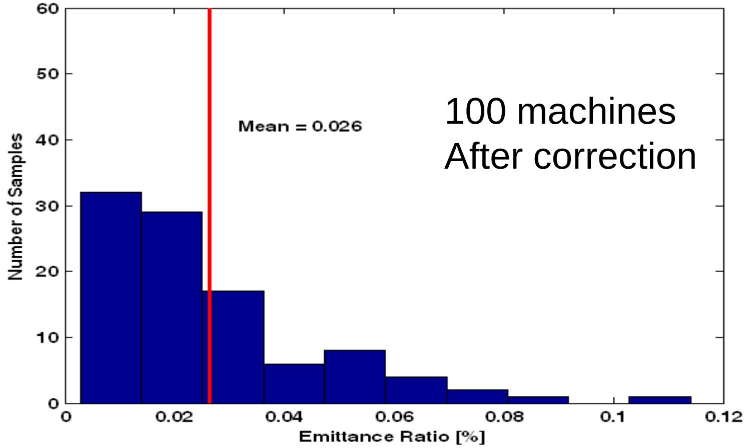
Dimension:
M: 16296 X 24 or 48
K: 24 or 48 skew quads
V: 16296 (96*168+168)
168 Monitors, 4 correctors per section, 96 in total.
Using SVD method to get K as wanted correction.

CORRECTION OF VERTICAL DISPERSION AND BETATRON COUPLING

Lattice: 79H2

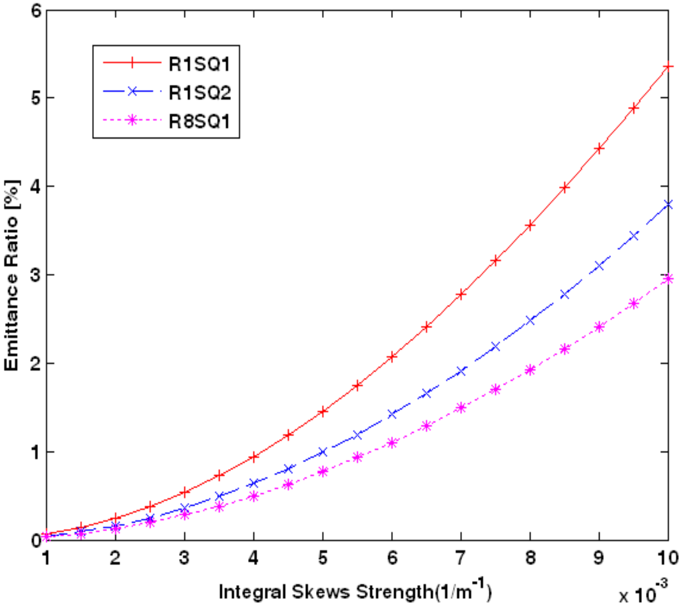
- **Using cross-plane response matrix and SVD method to correct both betatron coupling and vertical dispersion with a set of skew quadrupoles.**
- **With 48 skew quads, <1% emittance ratio can be achieved, and the maximum strength is < 5.4×10^{-3} m^{-1}**



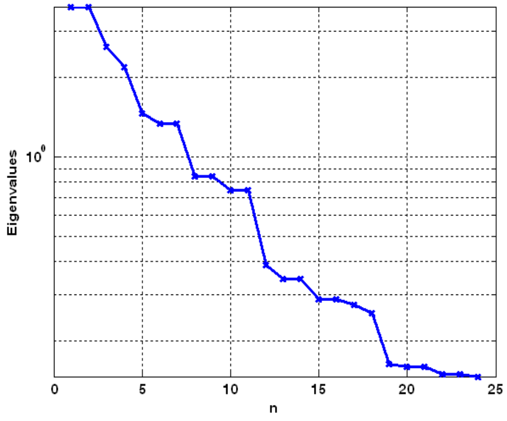


CORRECTION OF VERTICAL DISPERSION AND BETATRON COUPLING

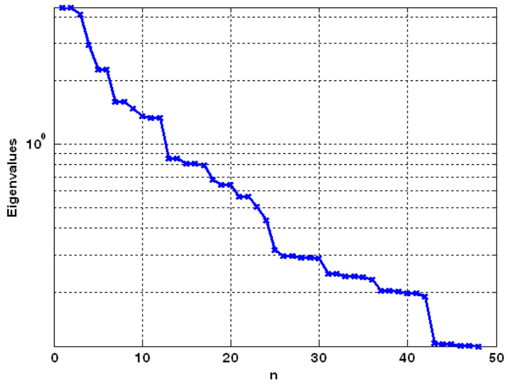
Lattice: 79H2



Increasing Coupling ratio Capability

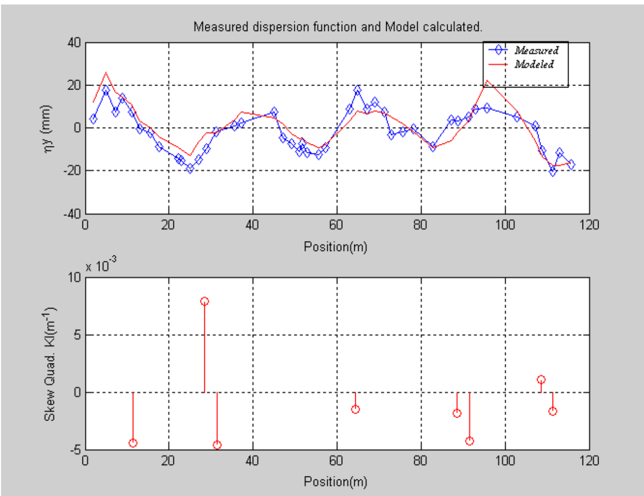


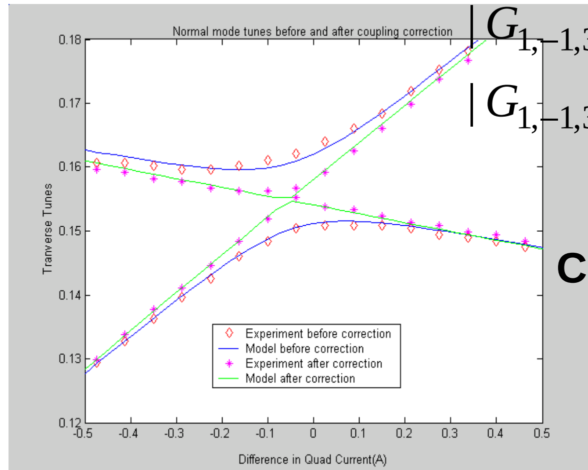
Eigenvalues of 24 Skew Quads.



Eigenvalues of 48 Skew Quads.

Experimental results (TLS)



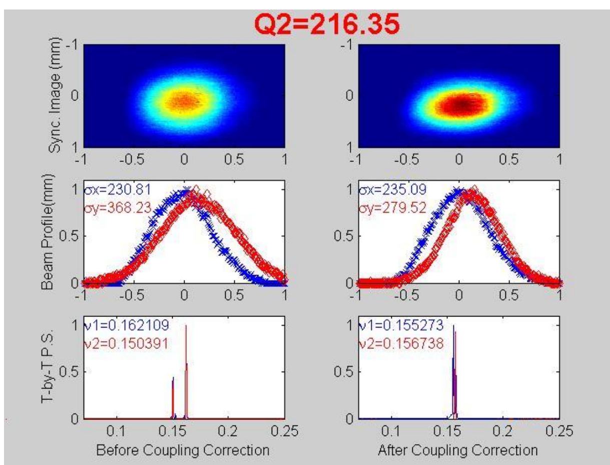


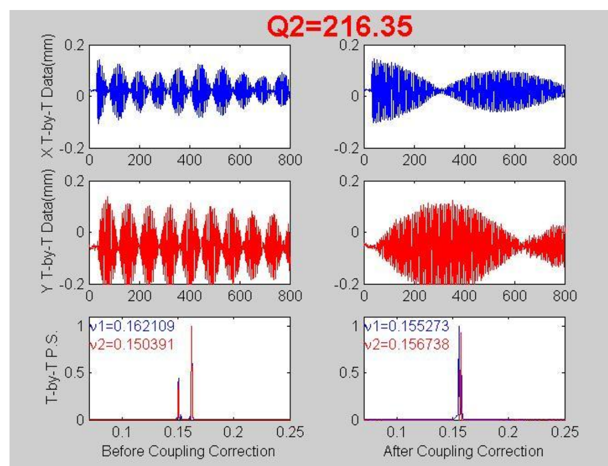
$|G_{1,-1,3}| = 0.0119$ before correction

$|G_{1,-1,3}| = 0.0016$ after correction

Coupling ratio is defined as:

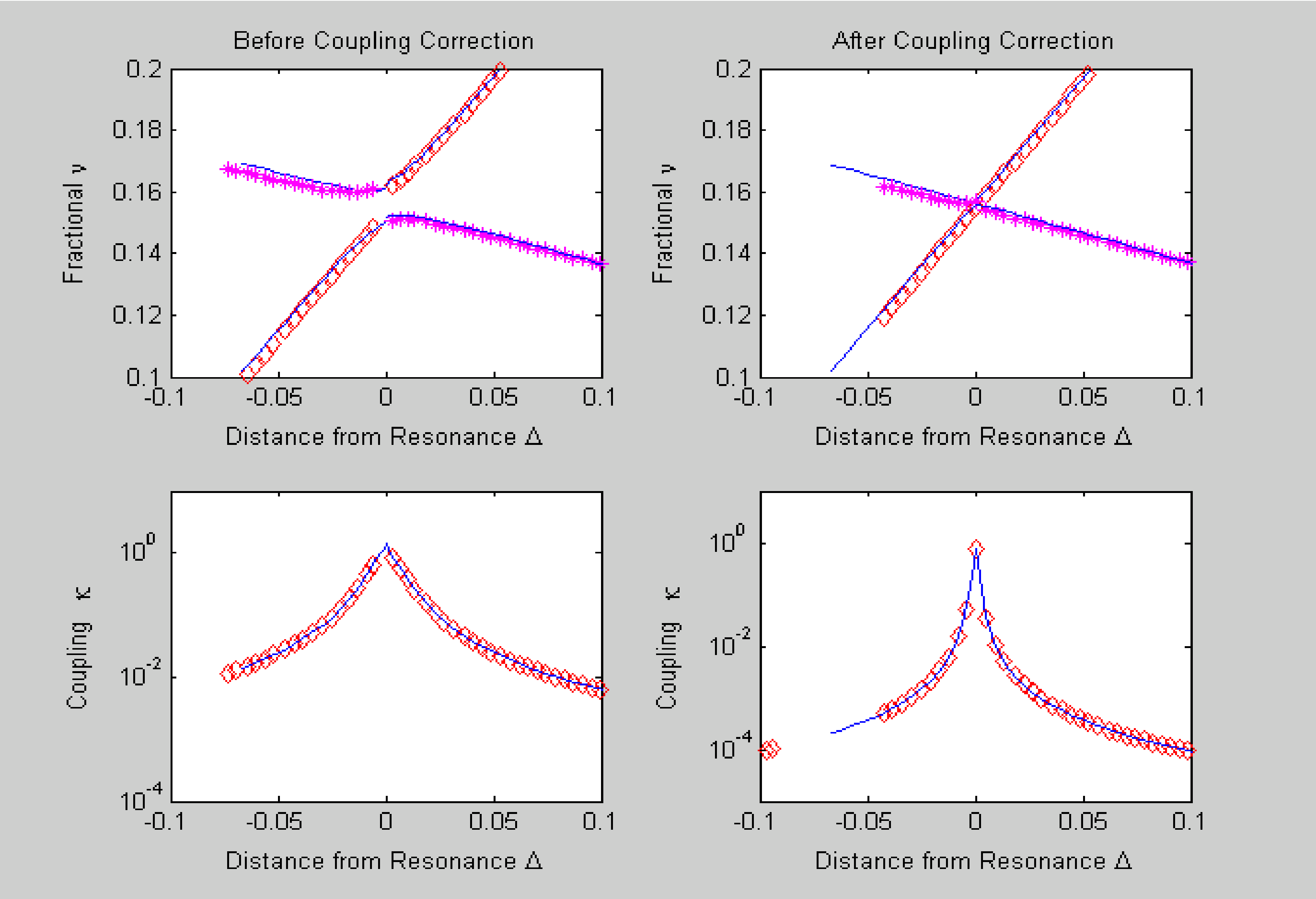
$$\kappa = \frac{G^2}{G^2 + 2\Delta^2}$$





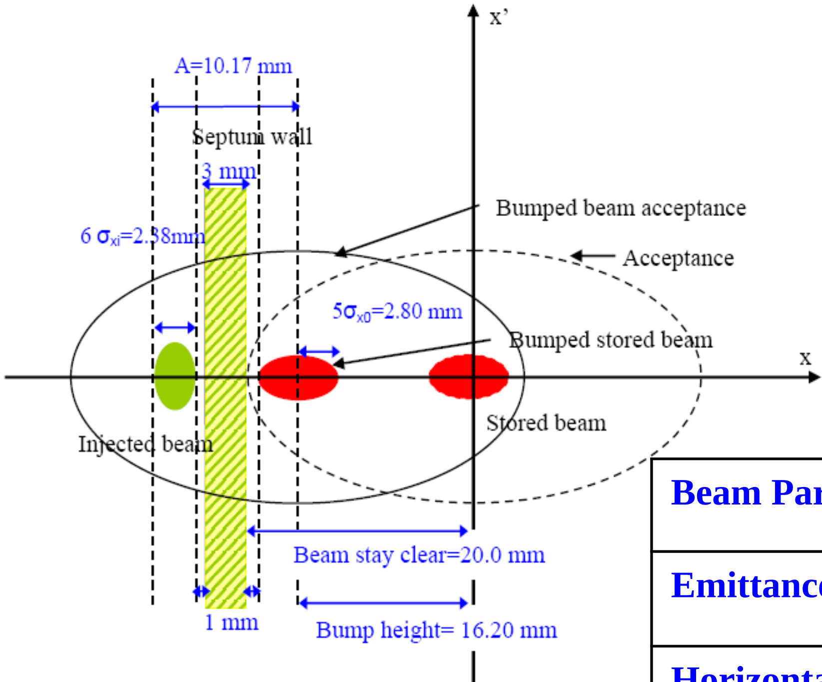
C.C. Kuo, et. al EPAC2002

TLS results



C.C. Kuo, et. al EPAC2002

Storage Ring injection scheme

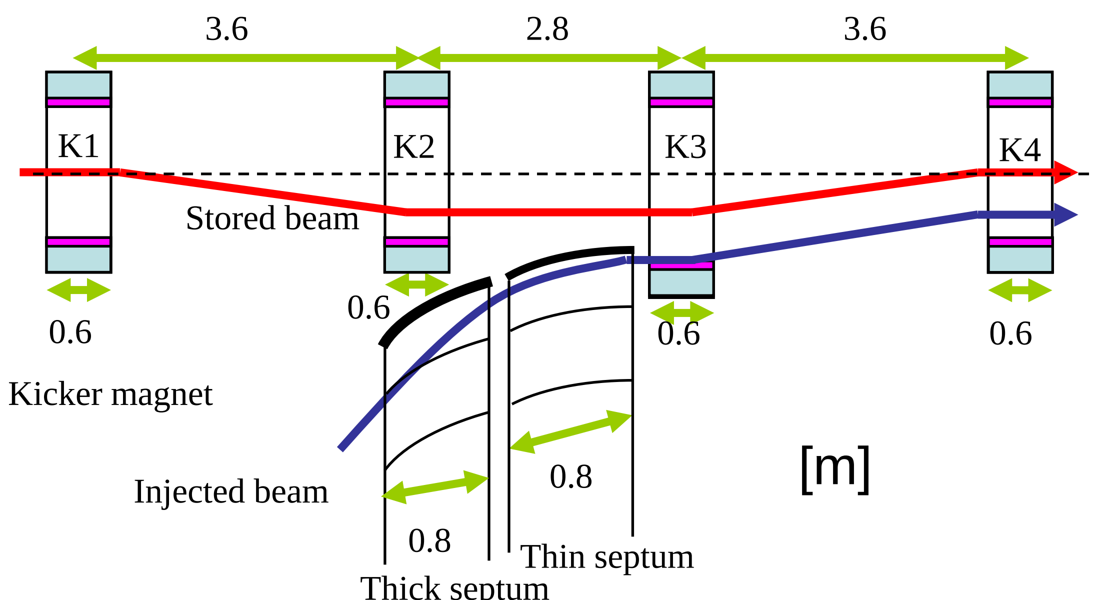


β_{xi} at injection point 2.5 m:

$$\beta_{xi}^{\ 2} + \frac{n\sigma_{x0} + T}{2\sqrt{\varepsilon_{xi}}}\,\beta_{xi}^{3/2} = \frac{\beta_{x0}^{2}}{2}$$

Beam Parameter	Storage Ring	Transfer Line
Emittance(3 GeV) (nm-rad)	1.912*4*4=30.6	31.3*2=62.6
Horizontal beta value at injection point (m)	10.211	2.503
Horizontal dispersion at injection point (m)	0.0	0.0
Energy spread	8.8E-4	9.5E-4
Horizontal beam size (mm)	0.559	0.396

Kicker and Septum Magnet Parameters for SR injection



	Thin septum (Pulse)& Thick septum (DC)	Kicker
Magnetic length (m)	0.8	0.60
Maximum field (T)	0.818	0.075
Deflection (mrad)	65.45	4.55
Pulse duration	500us (full-sine)	5.2us (half-sine)

Elliptical chamber size, Limited by bellows
Radii: 34mm (H), 15 mm (V)

79H2, +-4%

Off-momentum aperture, 79J2

$$x_k = \sqrt{A_{x,i}\beta_{x,k}} + \eta_{x,k}\delta$$

$$H_{x,i} = \gamma_{x,i}(\eta_{x,i}\delta)^2 + 2\alpha_{x,i}(\eta_{x,i}\delta)(\eta_{x,i}'\delta) + \beta_{x,i}(\eta_{x,i}'\delta)^2$$

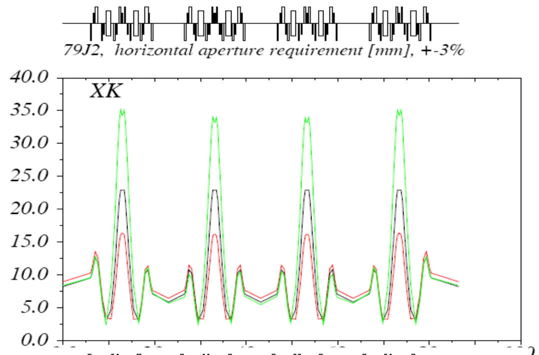
$$y_k = \sqrt{A_{y,i}\beta_{y,k}} + \eta_{y,k}\delta$$

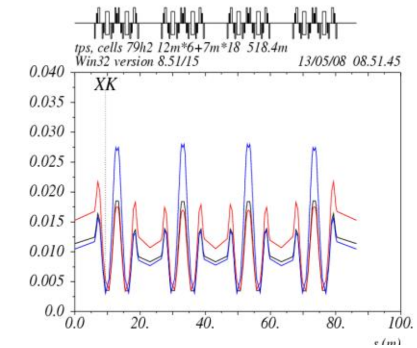
$$H_{y,i} = \gamma_{y,i}(\eta_{y,i}\delta)^2 + 2\alpha_{y,i}(\eta_{y,i}\delta)(\eta_{y,i}'\delta) + \beta_{y,i}(\eta_{y,i}'\delta)^2$$

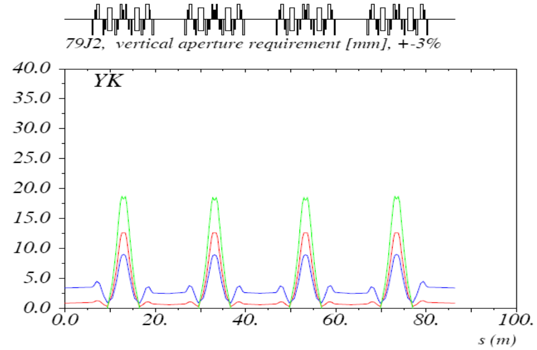
$$A_{x,i} = H_{x,i}\delta^2 + \kappa H_{y,i}\delta^2$$

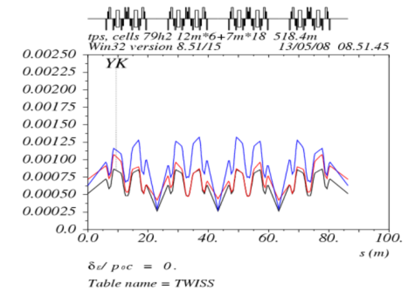
$$A_{y,i} = H_{y,i}\delta^2 + \kappa H_{x,i}\delta^2$$

Let $H_{y,i} = 0, \kappa = 1\%, \delta = +/-3\%$





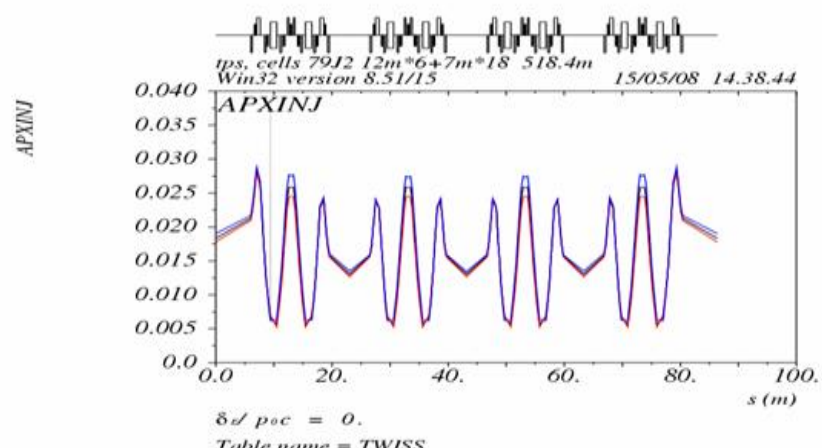




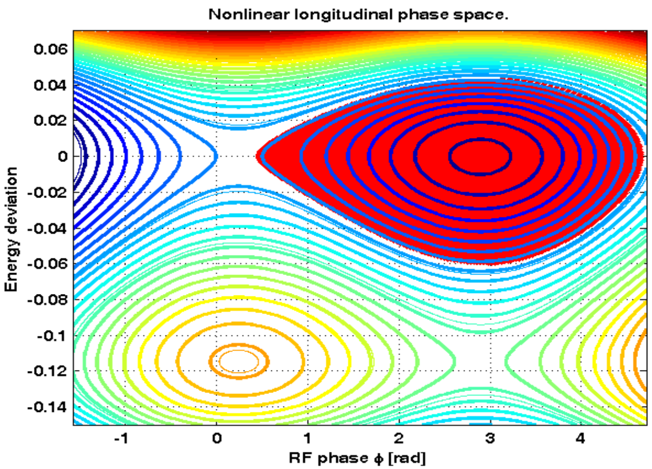
Injection aperture

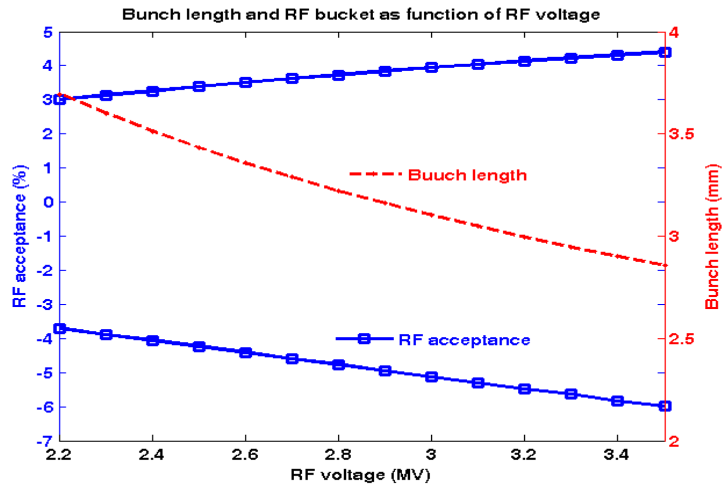
$$X_{injection} = (A_{x,\max}^2 \times \frac{\beta_{x0}}{\beta_{x,\max}} + (D_x\delta)^2)^{1/2}$$

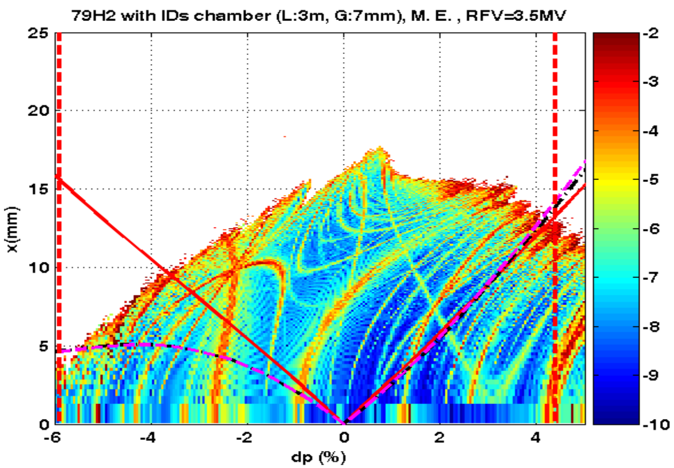
$$A_{x,\max} = 28.57mm, \delta = \pm 1\%$$

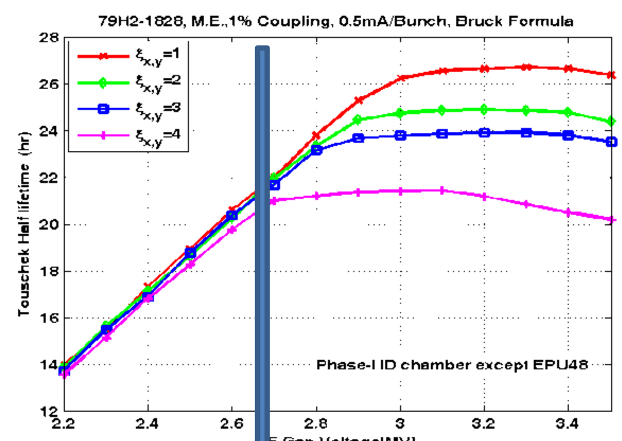


Energy acceptance and 6D tracking with chamber size





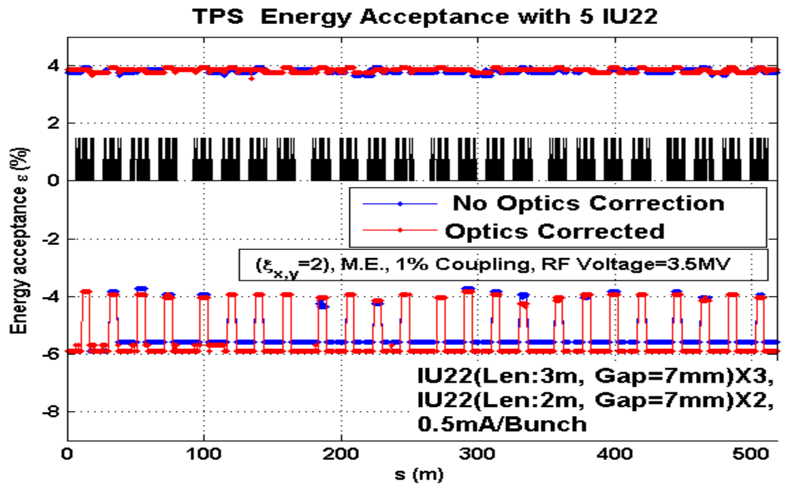


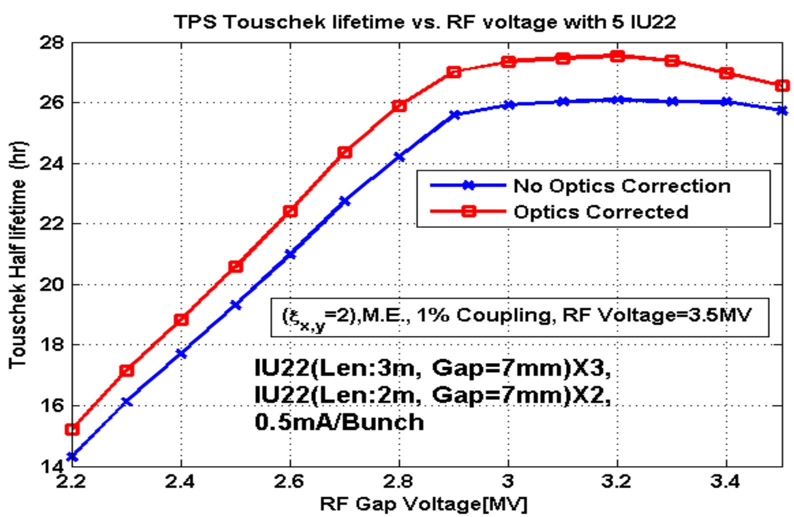


RF acceptance limited

Transverse aperture limited

Touschek Lifetime with 5 IU22





Bruck formula

$$\frac{1}{\tau_{T_{1/2}}} = \left(\frac{r_e^2 c N}{8\pi\gamma^3\sigma_l}\right)\cdot\frac{1}{L}\int_0^L \frac{C\left[\left(\frac{\varepsilon_{acc}(s)}{\gamma\sigma'_x(s)}\right)^2\right]}{\sigma_x(s)\sigma_z(s)\sigma'_x(s)\varepsilon_{acc}^2(s)}\,ds$$

$$C(\varsigma) = -1.5e^{-\varsigma} + \frac{\varsigma}{2}\int_\varsigma^\infty \frac{\ln u}{u}e^{-u}du + 0.5(3\varsigma - \varsigma\ln\varsigma + 2)\int_\varsigma^\infty \frac{e^{-u}}{u}du$$

Tracy-II 6-D tracking
- 5 IU22 gap 7 mm
- Multipole field errors
- 1% coupling
- Chromaticity=2
- Kick maps
- optics corrected

Radiation integral and electron beam properties

Radiation integral

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_3 = \oint \frac{1}{|\rho|^3} ds$$

$$I_4 = \oint \frac{\eta_x}{\rho} (\frac{1}{\rho^2} + 2K) ds$$

$$I_5 = \oint \frac{H}{|\rho|^3} ds,$$

$$\text{where } H = \frac{1}{\beta_x} (\eta_x^2 + (\alpha_x \eta_x + \beta_x \eta_x^{'})^2)$$

(1) Mom. comp. $\alpha_c = I_1 / 2\pi R$

(2) Emittance $\varepsilon_x = C_q \gamma^2 I_5 / (I_2 - I_4)$

$$C_q = (55/32\sqrt{3})\hbar / mc = 3.83 \times 10^{-13}$$

(3) Energy spread $(\sigma_E / E)^2 = C_q \gamma^2 I_3 / (2I_2 + I_4)$

(4)Energy loss per turn $U_0 \approx C_\gamma E^4 I_2 / 2\pi$

$$C_\gamma = (4\pi / 3) r_0 / (mc^2)^3 = 8.85 \times 10^{-5} m / (GeV)^3$$

(5) $J_x = 1 - I_4 / I_2, J_E = 2 + I_4 / I_2, D = I_4 / I_2$